Statistics for Business and Economics

Chapter 3

Probability

3.1

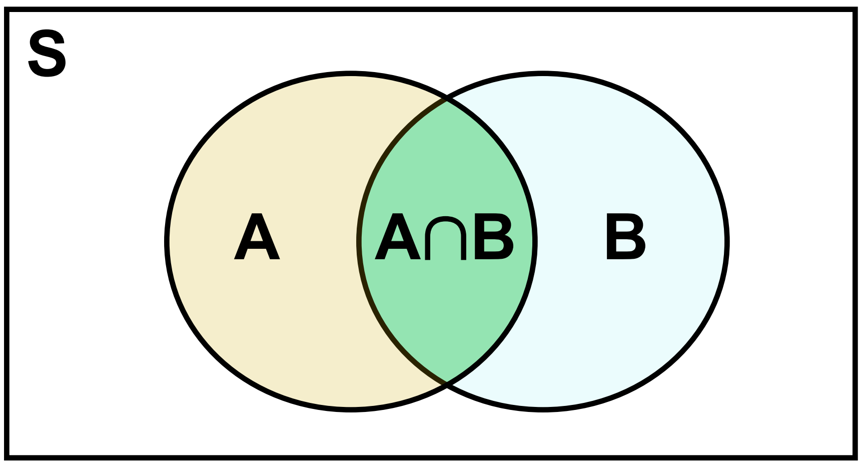
Important Terms

- Random Experiment – a process leading to an uncertain outcome
- Basic Outcome – a possible outcome of a random experiment
- Sample Space – the collection of all possible outcomes of a random experiment
- Event – any subset of basic outcomes from the sample space

Important Terms

(continued)

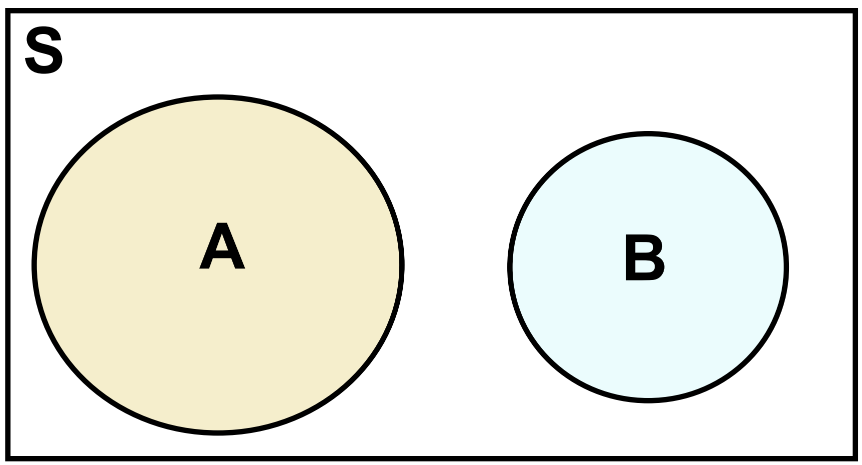
- Intersection of Events – If A and B are two events in a sample space S, then the intersection, A ∩ B, is the set of all outcomes in S that belong to both A and B



Important Terms

(continued)

- A and B are Mutually Exclusive Events if they have no basic outcomes in common
  - i.e., the set A ∩ B is empty

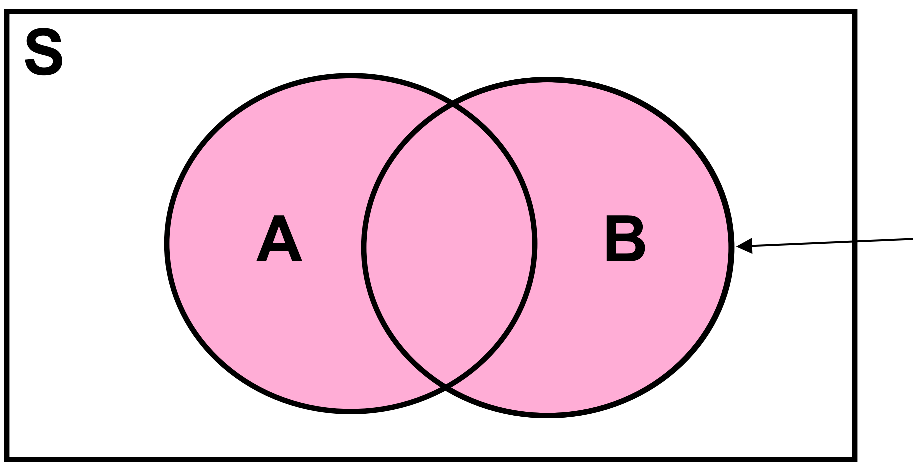


Important Terms

(continued)

- Union of Events – If A and B are two events in a sample space S, then the union, A ∪ B, is the set of all outcomes in S that belong to either

  A or B



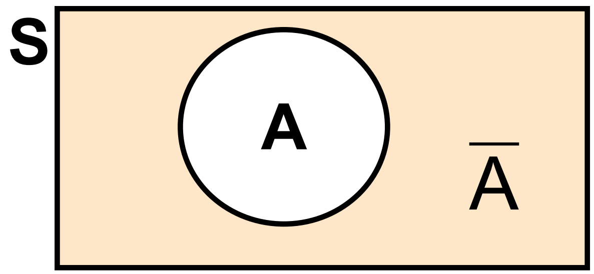
The entire shaded area represents A ∪ B

Important Terms

(continued)

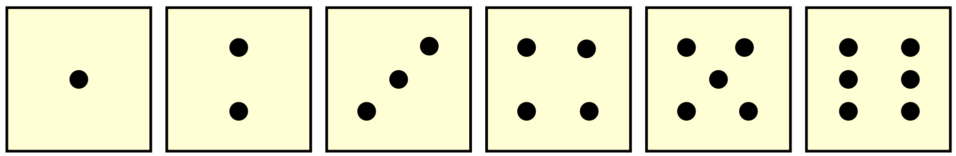
- Events E_1, E_2, … E_k are Collectively Exhaustive events if $E_1 \cup E_2 \cup . . . \cup E_k = S$
  - i.e., the events completely cover the sample space

- The Complement of an event A is the set of all basic outcomes in the sample space that do not belong to A. The complement is denoted $\overline{A}$



Example

Let the Sample Space be the collection of all possible outcomes of rolling one die:



S = [1, 2, 3, 4, 5, 6]

Let A be the event "Number rolled is even"

Let B be the event "Number rolled is at least 4"

Then

A = [2, 4, 6] and B = [4, 5, 6]

Examples

(continued)

S = [1, 2, 3, 4, 5, 6] | **A = [2, 4, 6]** | **B = [4, 5, 6]**

Complements:

$$\bar{A} = [1, 3, 5] \qquad \bar{B} = [1, 2, 3]$$

Intersections:

$$A \cap B = [4, 6] \qquad \bar{A} \cap B = [5]$$

Unions:

$$A \cup B = [2, 4, 5, 6]$$

$$A \cup \bar{A} = [1, 2, 3, 4, 5, 6] = S$$

Example

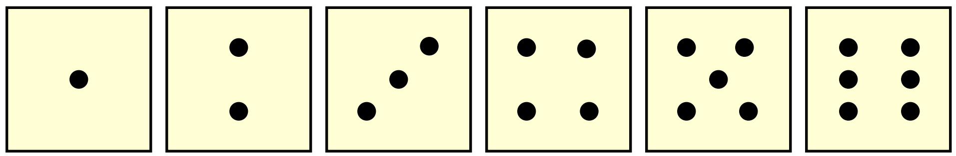
(continued)

S = [1, 2, 3, 4, 5, 6]	A = [2, 4, 6]	B = [4, 5, 6]

- Mutually exclusive:
  - A and B are not mutually exclusive
    - The outcomes 4 and 6 are common to both
- Collectively exhaustive:
  - A and B are not collectively exhaustive
    - $A \cup B$ does not contain 1 or 3

Another Example

- **What is Sample Space of rolling two dies?**



S = [(1,1), (1,2),…,(1,6),(2,1),(2,2), …,(2,6), (3,1),…,(3,6),(4,1),…,(5,1),…,(6,1),…,(6.6)]

- **Let A be the event “Both numbers are even”**

A = [(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)]

- **Let B be the event “Both numbers add to 3”**

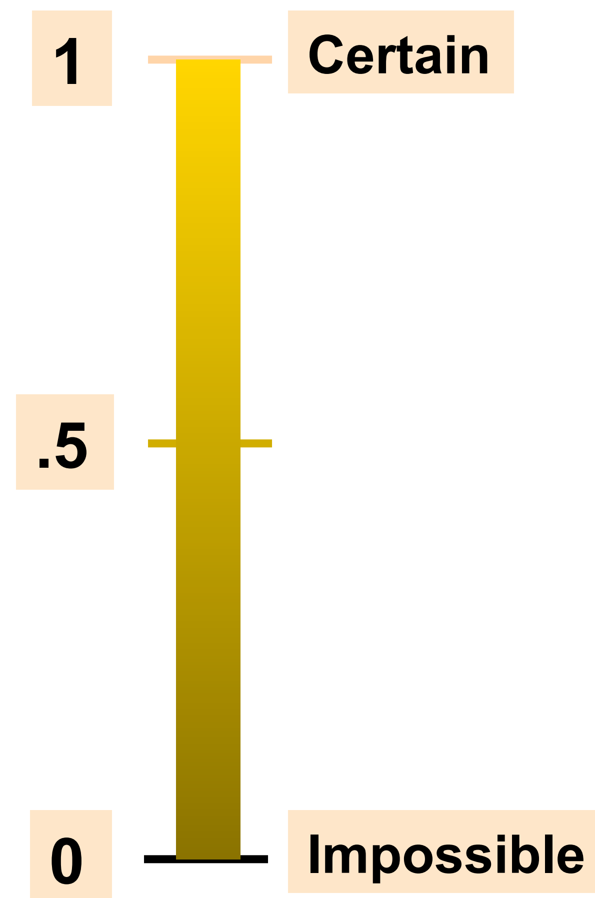
B = [(1,2),(2,1)]

3.2

Probability

- Probability – the chance that an uncertain event will occur (always between 0 and 1)

$0 \le P(A) \le 1$ For any event A



Assessing Probability

- There are three approaches to assessing the probability of an uncertain event:

  1. classical probability

$$\text{probability of event A} = \frac{N_A}{N} = \frac{\text{number of outcomes that satisfy the event}}{\text{total number of outcomes in the sample space}}$$

  - Assumes all outcomes in the sample space are equally likely to occur

Counting the Possible Outcomes

- Use the Combinations formula to determine the number of combinations of n things taken k at a time

$$C_k^n = \frac{n!}{k!(n-k)!}$$

- where
  - n! = n(n-1)(n-2)…(1)
  - 0! = 1 by definition

Example

- 5 candidates for 2 positions
- 3 candidates are men, 2 candidates are women
- Equal probability of hiring among 5 candidates

- What is the probability that no women will be hired?

Example

- The total number of possible combinations:

$$C_2^5 = \frac{5!}{2!(5-2)!} = \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{2\cdot 1\cdot (3\cdot 2\cdot 1)} = \frac{20}{2} = 10$$

- The number of possible combinations that both hired persons are men:

$$C_2^3 = \frac{3!}{2!(3-2)!} = \frac{3\cdot 2\cdot 1}{2\cdot 1\cdot (1)} = \frac{6}{2} = 3$$

- The probability that no women is hired:

3/10=30%

Assessing Probability

Three approaches (continued)

2. relative frequency probability

$$\text{probability of event A} = \frac{n_A}{n} = \frac{\text{number of events in the population that satisfy event A}}{\text{total number of events in the population}}$$

- the limit of the proportion of times that an event A occurs in a large number of trials, *n*

3. subjective probability

an individual opinion or belief about the probability of occurrence

Probability Postulates

1. If A is any event in the sample space S, then

$$0 \leq P(A) \leq 1$$

2. Let A be an event in S, and let O_i denote the basic outcomes. Then

$$P(A) = \sum_{A} P(O_i)$$

(the notation means that the summation is over all the basic outcomes in A)

3. $P(S) = 1$

3.3

Probability Rules

- The Complement rule:

$$P(\overline{A}) = 1 - P(A) \qquad \text{i.e., } P(A) + P(\overline{A}) = 1$$

- The Addition rule:
  - The probability of the union of two events is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

A Probability Table

Probabilities and joint probabilities for two events A and B are summarized in this table:

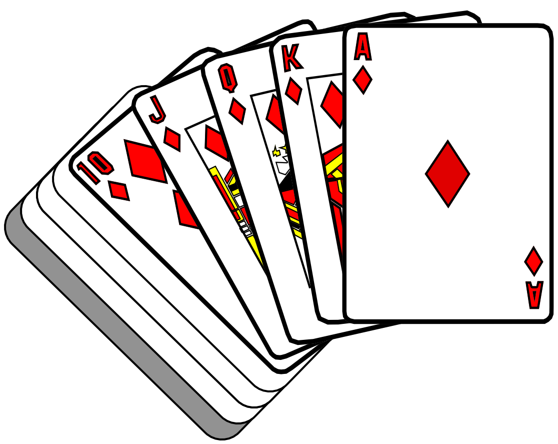
	B	$\overline{B}$	
A	$P(A \cap B)$	$P(A \cap \overline{B})$	$P(A)$
$\overline{A}$	$P(\overline{A} \cap B)$	$P(\overline{A} \cap \overline{B})$	$P(\overline{A})$
	$P(B)$	$P(\overline{B})$	$P(S) = 1.0$

Addition Rule Example

Consider a standard deck of 52 cards, with four suits: ♥ ♣ ♦ ♠

Let event A = card is an Ace

Let event B = card is from a red suit



Addition Rule Example

(continued)

$P(\text{Red} \cup \text{Ace}) = P(\text{Red}) + P(\text{Ace}) - P(\text{Red} \cap \text{Ace})$

$= 26/52 + 4/52 - 2/52 = 28/52$

Don't count the two red aces twice!

Type	Color		Total
	Red	Black	
Ace	2	2	4
Non-Ace	24	24	48
Total	26	26	52

Conditional Probability

- A conditional probability is the probability of one event, given that another event has occurred:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

→ **The conditional probability of A given that B has occurred**

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

→ **The conditional probability of B given that A has occurred**

Conditional Probability Example

- **Of the cars on a used car lot, 70% have air conditioning (AC) and 40% have a CD player (CD). 20% of the cars have both.**

- What is the probability that a car has a CD player, given that it has AC ?

  i.e., we want to find P(CD | AC)

Conditional Probability Example

(continued)

- Of the cars on a used car lot, 70% have air conditioning (AC) and 40% have a CD player (CD). 20% of the cars have both.

	CD	No CD	Total
AC	.2	.5	.7
No AC	.2	.1	.3
Total	.4	.6	1.0

$$P(CD \mid AC) = \frac{P(CD \cap AC)}{P(AC)} = \frac{.2}{.7} = .2857$$

Conditional Probability Example

(continued)

- **Given AC, we only consider the top row (70% of the cars). Of these, 20% have a CD player. 20% of 70% is 28.57%.**

	CD	No CD	Total
AC	.2	.5	.7
No AC	.2	.1	.3
Total	.4	.6	1.0

$$P(CD \mid AC) = \frac{P(CD \cap AC)}{P(AC)} = \frac{.2}{.7} = .2857$$

Multiplication Rule

- Multiplication rule for two events A and B:

$$P(A \cap B) = P(A|B)P(B)$$

- also

$$P(A \cap B) = P(B|A)P(A)$$

Multiplication Rule Example

$$P(\text{Red} \cap \text{Ace}) = P(\text{Red} \mid \text{Ace})P(\text{Ace})$$

$$= \left(\frac{2}{4}\right)\left(\frac{4}{52}\right) = \frac{2}{52}$$

$$= \frac{\text{number of cards that are red and ace}}{\text{total number of cards}} = \frac{2}{52}$$

Type	Color		Total
	Red	Black	
Ace	2	2	4
Non-Ace	24	24	48
Total	26	26	52

Statistical Independence

- Two events are statistically independent if and only if:

$$P(A \cap B) = P(A)P(B)$$

  - Events A and B are independent when the probability of one event is not affected by the other event

- If A and B are independent, then

$$P(A \mid B) = P(A) \qquad \textbf{if } P(B)>0$$

$$P(B \mid A) = P(B) \qquad \textbf{if } P(A)>0$$

Statistical Independence Example

- Of the cars on a used car lot, 70% have air conditioning (AC) and 40% have a CD player (CD). 20% of the cars have both.

	CD	No CD	Total
AC	.2	.5	.7
No AC	.2	.1	.3
Total	.4	.6	1.0

- Are the events AC and CD statistically independent?

Statistical Independence Example

(continued)

	CD	No CD	Total
AC	.2	.5	.7
No AC	.2	.1	.3
Total	.4	.6	1.0

$P(AC \cap CD) = 0.2$

$P(AC) = 0.7$

$P(CD) = 0.4$

$P(AC)P(CD) = (0.7)(0.4) = 0.28$

$P(AC \cap CD) = 0.2 \neq P(AC)P(CD) = 0.28$

So the two events are not statistically independent

Bivariate Probabilities

3.4

Outcomes for bivariate events:

	B_1	B_2	. . .	B_k
A_1	$P(A_1 \cap B_1)$	$P(A_1 \cap B_2)$	. . .	$P(A_1 \cap B_k)$
A_2	$P(A_2 \cap B_1)$	$P(A_2 \cap B_2)$	. . .	$P(A_2 \cap B_k)$
. . .	. . .	. . .	. . .	. . .
A_h	$P(A_h \cap B_1)$	$P(A_h \cap B_2)$	. . .	$P(A_h \cap B_k)$

Joint Distribution of X and Y

- Consider two random variables: X and Y
- X takes n possible values:

$$\{x_1,x_2,\ldots,x_n\}$$

- Y takes m possible values:

$$\{y_1,y_2,\ldots,y_m\}$$

- Joint Distribution of X and Y can be described by Bivariate probabilities.

3.4

Distribution of (X,Y)

	X=1	X=2	. . .	X=n
Y=1	P(X=x_1,Y=y_1)	P(X=x_2,Y=y_1)	. . .	P(X=x_n,Y=y_1)
Y=2	P(X=x_1,Y=x_2)	P(X=x_2,Y=y_2)	. . .	P(X=x_n,Y=y_2)
. . .	. . .	. . .	. . .	. . .
Y=m	P(X=x_1,Y=y_m)	P(X=x_2,Y=y_m)	. . .	P(X=x_n,Y=y_m)

Joint and Marginal Probabilities

- The probability of a joint event, A ∩ B:

$$P(A \cap B) = \frac{\text{number of outcomes satisfying A and B}}{\text{total number of elementary outcomes}}$$

- Computing a marginal probability:

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_k)$$

  - Where B_1, B_2, …, B_k are k mutually exclusive and collectively exhaustive events

Marginal Probability Example

P(Ace)

$$= P(\text{Ace} \cap \text{Red}) + P(\text{Ace} \cap \text{Black}) = \frac{2}{52} + \frac{2}{52} = \frac{4}{52}$$

Type	Color		Total
	Red	Black	
Ace	2	2	4
Non-Ace	24	24	48
Total	26	26	52

3.5

Bayes' Theorem

$$P(E_i \mid A) = \frac{P(A \mid E_i)P(E_i)}{P(A)}$$

$$= \frac{P(A \mid E_i)P(E_i)}{P(A \mid E_1)P(E_1) + P(A \mid E_2)P(E_2) + \ldots + P(A \mid E_k)P(E_k)}$$

- where:

  E_i = i^{th} event of k mutually exclusive and collectively exhaustive events

  A = new event that might impact $P(E_i)$

Bayes' Theorem Example

- If a person has the disease (D+), a blood test is positive (T+) with 95% probability. If a person is free of the disease (D-), the test comes back negative (T-) with 90% probability.

$$P(T+|D+) = 0.95 \quad \text{and} \quad P(T-|D-) = 0.90$$

- 1% people have the disease: P(D+) = 0.01.
- What is the probability that you have the disease when your blood test is positive?

Bayes' Theorem Example

(continued)

- What is the probability that you have the disease if your blood test is positive?
- Let D+ = disease, D- = no disease

  T+ = positive test, T- = negative test
- P(D+) = .01, P(D-) = 1-P(D+) = .99
- P(T+|D+) = .95, P(T-|D-) = .90
- Goal is to find P(D+|T+)

Bayes' Theorem Example

(continued)

Apply Bayes' Theorem:

$$P(D+|T+) = \frac{P(T+|D+)P(D+)}{P(T+|D+)P(D+) + P(T+|D-)P(D-)}$$

$$= \frac{(.95)(.01)}{(.95)(.01) + (1 - .90)(.99)}$$

$$= \frac{.0095}{.0095 + .099} = .08756$$

So the revised probability of having disease is 8.76 percent!

Chapter Summary

- Defined basic probability concepts
  - Sample spaces and events, intersection and union of events, mutually exclusive and collectively exhaustive events, complements
- Examined basic probability rules
  - Complement rule, addition rule, multiplication rule
- Defined conditional, joint, and marginal probabilities
- Defined statistical independence
- Discussed Bayes' theorem