Statistics for Business and Economics

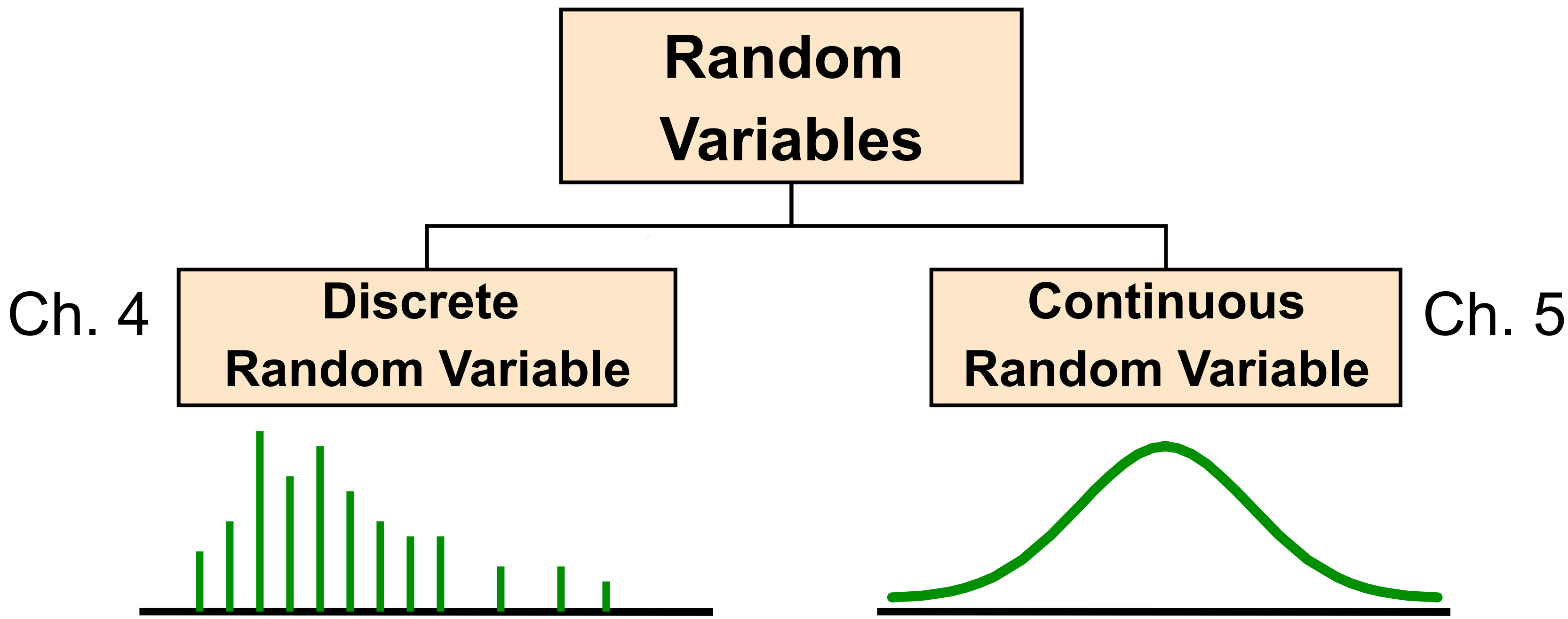
Chapter 4

Discrete Random Variables and Probability Distributions

4.1

Introduction to Probability Distributions

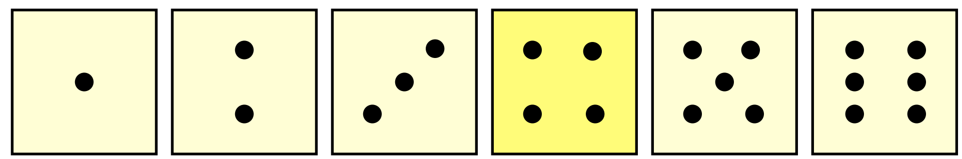
- **Random Variable**
  - Represents a possible numerical value from a random experiment



Discrete Random Variables

- Can only take on a countable number of values

Examples:



- **Roll a die twice**
  Let X be the number of times 4 comes up
  (then X could be 0, 1, or 2 times)

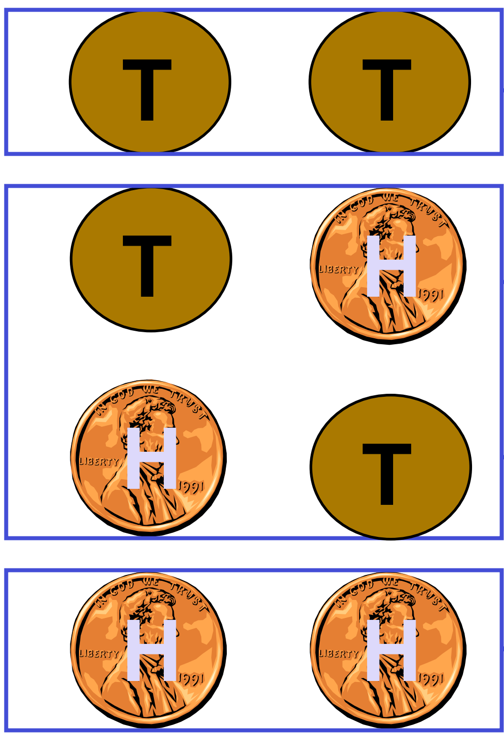
- **Toss a coin 3 times.**
  Let X be the number of heads
  (then X = 0, 1, 2, or 3)

4.2 Discrete Probability Distribution

Experiment: Toss 2 Coins. Let X = # heads.

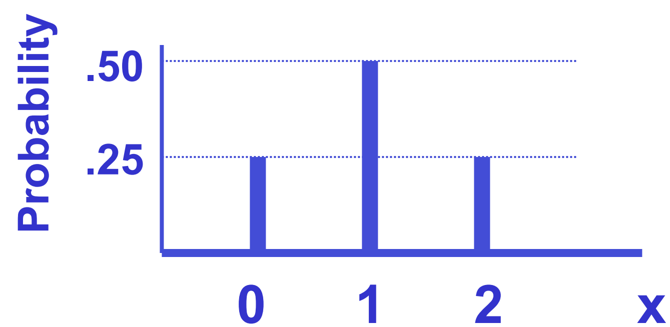
Show P(x), i.e., P(X = x), for all values of x:

4 possible outcomes



Probability Distribution

x Value	Probability
0	1/4 = .25
1	2/4 = .50
2	1/4 = .25



Probability Distribution Required Properties

4.3

- $P(x) \geq 0$ for any value of x

- The individual probabilities sum to 1;

$$\sum_x P(x) = 1$$

(The notation indicates summation over all possible x values)

Cumulative Probability Function

- The cumulative probability function, denoted $F(x_0)$, shows the probability that X is less than or equal to x_0

$$F(x_0) = P(X \le x_0)$$

- In other words,

$$F(x_0) = \sum_{x \le x_0} P(x)$$

Expected Value

- Expected Value (or mean) of a discrete distribution (Weighted Average)

$$\mu = E(X) = \sum_{x} xP(x)$$

- Example: Toss 2 coins,
  x = # of heads,
  compute expected value of X:

x	P(x)
0	.25
1	.50
2	.25

E(X) = (0 x .25) + (1 x .50) + (2 x .25)
= 1.0

Variance and Standard Deviation

- Variance of a discrete random variable X

$$\sigma^2 = E(X-\mu)^2 = \sum_x (x-\mu)^2 P(x)$$

- Standard Deviation of a discrete random variable X

$$\sigma = \sqrt{\sigma^2} = \sqrt{\sum_x (x-\mu)^2 P(x)}$$

Standard Deviation Example

- Example: Toss 2 coins, X = # heads, compute standard deviation (recall E(x) = 1)

$$\sigma = \sqrt{\sum_x (x-\mu)^2 P(x)}$$

$$\sigma = \sqrt{(0-1)^2(.25)+(1-1)^2(.50)+(2-1)^2(.25)} = \sqrt{.50} = .707$$

Possible number of heads = 0, 1, or 2

Functions of Random Variables

- If $P(x)$ is the probability function of a discrete random variable X , and $g(X)$ is some function of X , then the expected value of function g is

$$E[g(X)] = \sum_{x} g(x)P(x)$$

Linear Functions of Random Variables

- Let a and b be any constants.

- a) $E(a) = a \quad\quad \text{and} \quad\quad Var(a) = 0$

  i.e., if a random variable always takes the value a, it will have mean a and variance 0

- b) $E(bX) = bE(X) \quad\quad \text{and} \quad\quad Var(bX) = b^2 Var(X)$

  i.e., the expected value of b·X is b·E(X)

Linear Functions of Random Variables

(continued)

- Let random variable X have mean μ_x and variance σ^2_x
- Let a and b be any constants.
- Let Y = a + bX
- Then the mean and variance of Y are

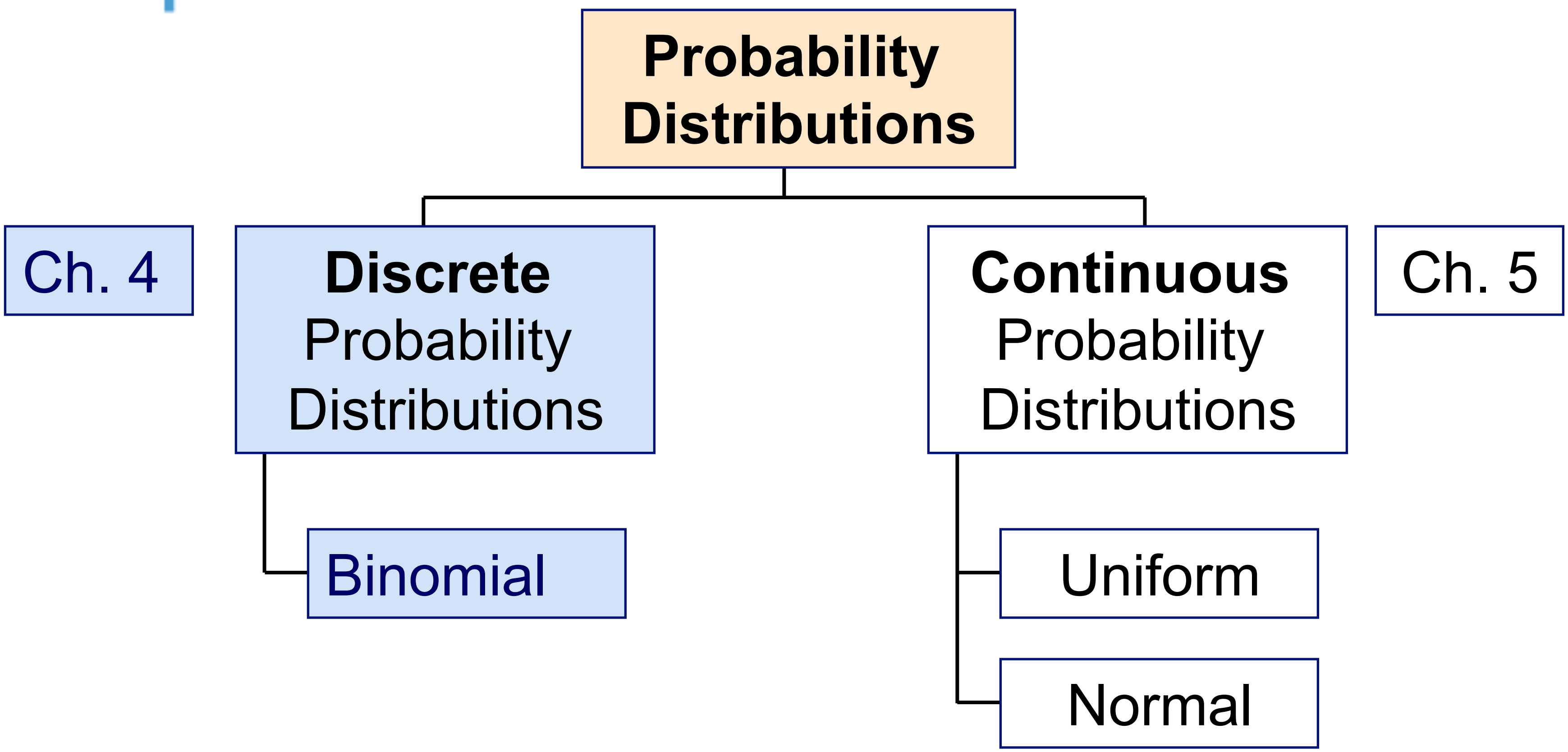
$$E(Y) = E(a + bX) = a + bE(X)$$

$$Var(Y) = Var(a + bX) = b^2 Var(X)$$

- so that the standard deviation of Y is

$$\sigma_Y = |b|\sigma_X$$

Probability Distributions



Bernoulli Distribution

- Consider only two outcomes: "success" or "failure"
- Let p denote the probability of success
- Let 1 – p be the probability of failure
- Define random variable X:

  X = 1 if success, X = 0 if failure

- Then the Bernoulli probability function is

$$P(X = 0) = (1 - p) \quad \text{and} \quad P(X = 1) = p$$

Bernoulli Distribution Mean and Variance

- The mean is $\mu = p$

$$\mu = E(X) = \sum_{X=0,1} xP(X = x) = (0)(1-p) + (1)p = p$$

- The variance is $\sigma^2 = p(1 - p)$

$$\sigma^2 = E[(X-\mu)^2] = \sum_{X=0,1} (x-\mu)^2 P(X = x)$$

$$= (0-p)^2(1-p) + (1-p)^2 p = p(1-p)$$

Sequences of x Successes in n Trials

- The number of sequences with x successes in n independent trials is:

$$C_x^n = \frac{n!}{x!(n-x)!}$$

Where $n! = n\cdot(n-1)\cdot(n-2)\cdot \ldots \cdot 1$ and $0! = 1$

- These sequences are mutually exclusive, since no two can occur at the same time

Binomial Probability Distribution

- A fixed number of observations, n
  - e.g., 15 tosses of a coin; ten light bulbs taken from a warehouse
- Two mutually exclusive and collectively exhaustive categories
  - e.g., head or tail in each toss of a coin; defective or not defective light bulb
  - Generally called "success" and "failure"
  - Probability of success is p , probability of failure is 1 – p
- Constant probability for each observation
  - e.g., Probability of getting a tail is the same each time we toss the coin
- Observations are independent
  - The outcome of one observation does not affect the outcome of the other

Possible Binomial Distribution Settings

- A manufacturing plant labels items as either defective or acceptable
- A firm bidding for contracts will either get a contract or not
- A marketing research firm receives survey responses of “yes I will buy” or “no I will not”
- New job applicants either accept the offer or reject it

Binomial Distribution Formula

$$P(X=x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

P(x) = probability of **x** successes in **n** trials,
with probability of success **p** on each trial

x = number of 'successes' in sample,
(x = 0, 1, 2, ..., n)

n = sample size (number of trials or observations)

p = probability of "success"

Example: Flip a coin four times, let x = # heads:

n = 4

p = 0.5

1 - p = (1 - 0.5) = 0.5

x = 0, 1, 2, 3, 4

Example: Calculating a Binomial Probability

What is the probability of one success in five observations if the probability of success is 0.1?

X = 1, n = 5, and p = 0.1

$$P(X = 1) = \frac{n!}{x!(n-x)!} p^X (1-p)^{n-X}$$

$$= \frac{5!}{1!(5-1)!} (0.1)^1 (1-0.1)^{5-1}$$

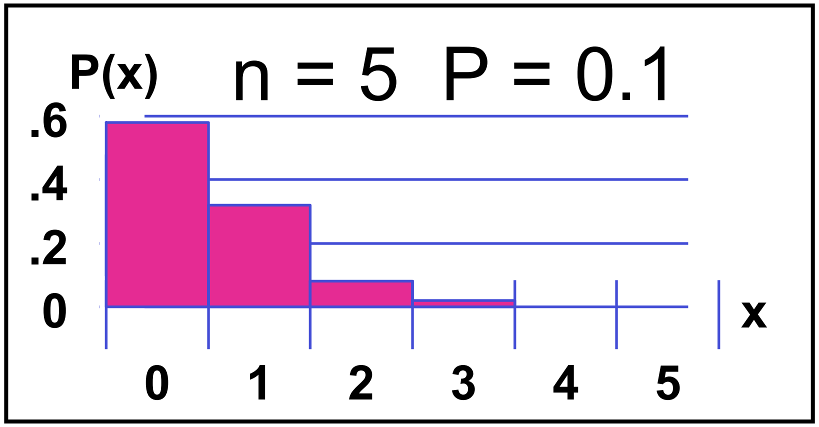
$$= (5)(0.1)(0.9)^4$$

$$= .32805$$

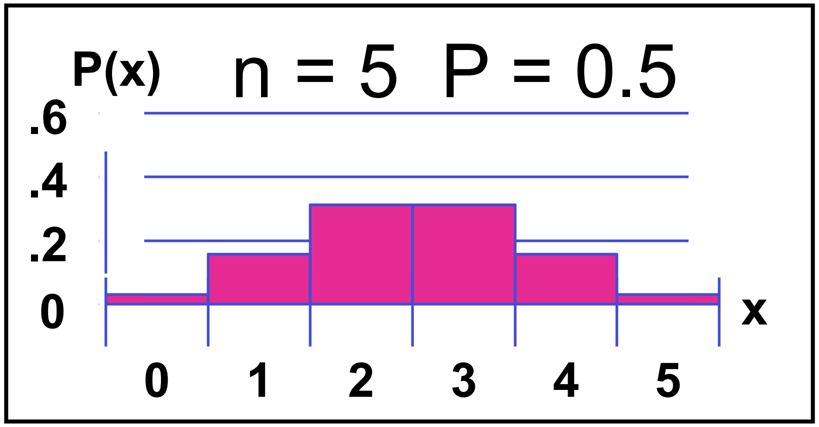
Binomial Distribution

- The shape of the binomial distribution depends on the values of P and n

  - Here, n = 5 and P = 0.1



  - Here, n = 5 and P = 0.5



Binomial Distribution Mean and Variance

- Mean

$$\mu = E(X) = np$$

- Variance and Standard Deviation

$$\sigma^2 = np(1-p)$$

$$\sigma = \sqrt{np(1-p)}$$

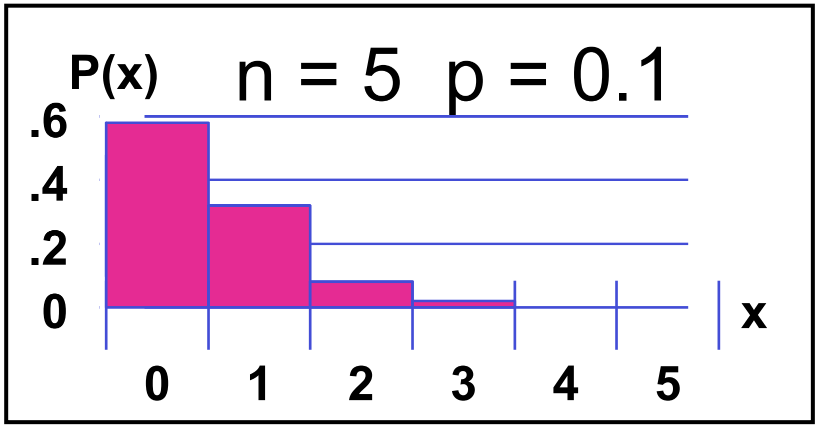
Where n = sample size
p = probability of success
(1 – p) = probability of failure

Binomial Characteristics

Examples

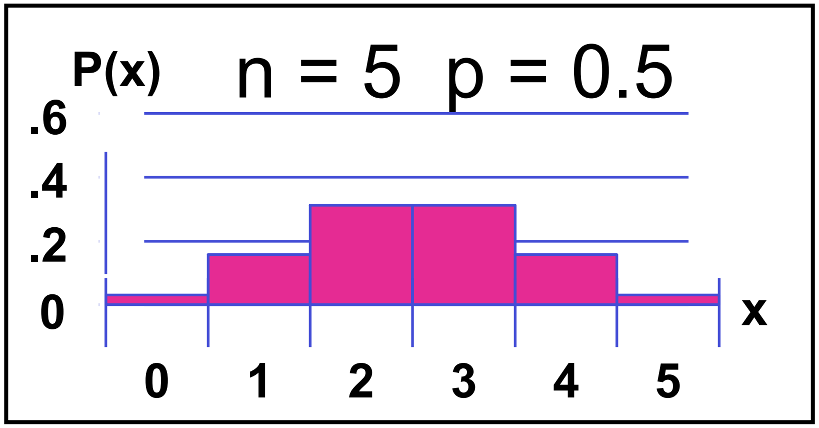
$$\mu = np = (5)(0.1) = 0.5$$

$$\sigma = \sqrt{np(1-p)} = \sqrt{(5)(0.1)(1-0.1)} = 0.6708$$



$$\mu = np = (5)(0.5) = 2.5$$

$$\sigma = \sqrt{np(1-p)} = \sqrt{(5)(0.5)(1-0.5)} = 1.118$$



Using Binomial Tables

N	x	…	p=.20	p=.25	p=.30	p=.35	p=.40	p=.45	p=.50
10	0	…	0.1074	0.0563	0.0282	0.0135	0.0060	0.0025	0.0010
	1	…	0.2684	0.1877	0.1211	0.0725	0.0403	0.0207	0.0098
	2	…	0.3020	0.2816	0.2335	0.1757	0.1209	0.0763	0.0439
	3	…	0.2013	0.2503	0.2668	**0.2522**	0.2150	0.1665	0.1172
	4	…	0.0881	0.1460	0.2001	0.2377	0.2508	0.2384	0.2051
	5	…	0.0264	0.0584	0.1029	0.1536	0.2007	0.2340	0.2461
	6	…	0.0055	0.0162	0.0368	0.0689	0.1115	0.1596	0.2051
	7	…	0.0008	0.0031	0.0090	0.0212	0.0425	0.0746	0.1172
	8	…	0.0001	0.0004	0.0014	0.0043	0.0106	**0.0229**	0.0439
	9	…	0.0000	0.0000	0.0001	0.0005	0.0016	0.0042	0.0098
	10	…	0.0000	0.0000	0.0000	0.0000	0.0001	0.0003	0.0010

Examples:

n = 10, X = 3, p = 0.35: P(X = 3|n =10, p = 0.35) = .2522

n = 10, X = 8, p = 0.45: P(X = 8|n =10, p = 0.45) = .0229

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Joint Probability Functions

- A joint probability function is used to express the probability that X takes the specific value x and simultaneously Y takes the value y, as a function of x and y

$$P(x,y) = P(X = x \cap Y = y)$$

- The marginal probabilities are

$$P(x) = \sum_{y} P(x,y)$$

$$P(y) = \sum_{x} P(x,y)$$

Conditional Probability Functions

- The conditional probability function of the random variable Y expresses the probability that Y takes the value y when the value x is specified for X.

$$P(y|x)=\frac{P(x,y)}{P(x)}$$

- Similarly, the conditional probability function of X, given Y = y is:

$$P(x|y)=\frac{P(x,y)}{P(y)}$$

Independence

- The jointly distributed random variables X and Y are said to be independent if and only if their joint probability function is the product of their marginal probability functions:

$$P(x, y) = P(x)P(y)$$

for all possible pairs of values x and y

- A set of k random variables are independent if and only if

$$P(x_1, x_2, \cdots, x_k) = P(x_1)P(x_2)\cdots P(x_k)$$

Conditional Mean and Variance

- The conditional mean is

$$\mu_{Y|X} = E[Y \mid X] = \sum_{Y} y\, P(y \mid x)$$

- E[Y|X] is a function of X and, therefore, is also called as ``the conditional expectation function (CEF)''

- The conditional variance is

$$\sigma^2_{Y|X} = E[(Y - \mu_{Y|X})^2 \mid X] = \sum_{Y} (y - \mu_{Y|X})^2 P(y \mid x)$$

Covariance

- Let X and Y be discrete random variables with means μ_X and μ_Y
- The expected value of $(X - \mu_X)(Y - \mu_Y)$ is called the covariance between X and Y
- For discrete random variables

$$\text{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = \sum_x \sum_y (x-\mu_X)(y-\mu_Y)P(x,y)$$

- An equivalent expression is

$$\text{Cov}(X,Y) = E(XY) - \mu_X\mu_Y = \sum_x \sum_y xyP(x,y) - \mu_X\mu_Y$$

Covariance and Independence

- The covariance measures the strength of the linear relationship between two variables

- If two random variables are statistically independent, the covariance between them is 0
  - The converse is not necessarily true

Correlation

- The correlation between X and Y is:

$$\rho = \text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

- $\rho = 0$: no linear relationship between X and Y
- $\rho > 0$: positive linear relationship between X and Y
  - when X is high (low) then Y is likely to be high (low)
  - $\rho = +1$: perfect positive linear dependency
- $\rho < 0$: negative linear relationship between X and Y
  - when X is high (low) then Y is likely to be low (high)
  - $\rho = -1$: perfect negative linear dependency

Portfolio Analysis

- Let random variable X be the share price for stock A
- Let random variable Y be the share price for stock B
- The market value, W, for the portfolio is given by the linear function

$$W = aX + bY$$

- ``a'' and ``b'' are the numbers of shares of stock A and B, respectively.
- The return from holding the portfolio W:

$$\Delta W = a\Delta X + b\Delta Y$$

Portfolio Analysis

(continued)

- The mean value for ΔW is

$$E[\Delta W] = E[a\Delta X + b\Delta Y]$$
$$= aE[\Delta X] + bE[\Delta Y]$$

- The variance for ΔW is

$$\sigma^2_{\Delta W} = a^2\sigma^2_{\Delta X} + b^2\sigma^2_{\Delta Y} + 2ab\mathrm{Cov}(\Delta X, \Delta Y)$$

or using the correlation formula

$$\sigma^2_{\Delta W} = a^2\sigma^2_{\Delta X} + b^2\sigma^2_{\Delta Y} + 2ab\mathrm{Corr}(\Delta X, \Delta Y)\sigma_{\Delta X}\sigma_{\Delta Y}$$

Example: Investment Returns

Return per $100 for two types of investments

		Investment	
P(ΔX,ΔY)	**Economic condition**	**Bond Fund X**	**Aggressive Fund Y**
0.2	Recession	+ $ 7	- $20
0.5	Stable Economy	+ 4	+ 6
0.3	Expanding Economy	+ 2	+ 35

$$E(\Delta X) = (7)(.2) + (4)(.5) + (2)(.3) = 4$$

$$E(\Delta Y) = (-20)(.2) + (6)(.5) + (35)(.3) = 9.5$$

Computing the Standard Deviation for Investment Returns

P(ΔX,ΔY)	Economic condition	Investment: Bond Fund X	Investment: Aggressive Fund Y
0.2	Recession	+ $ 7	- $20
0.5	Stable Economy	+ 4	+ 6
0.3	Expanding Economy	+ 2	+ 35

$$\sigma_{\Delta X} = \sqrt{\mathrm{Var}(\Delta X)} = \sqrt{(7-4)^2(0.2) + (4-4)^2(0.5) + (2-4)^2(0.3)}$$
$$= 1.73$$

$$\sigma_{\Delta Y} = \sqrt{\mathrm{Var}(\Delta Y)} = \sqrt{(-20-9.5)^2(0.2) + (6-9.5)^2(0.5) + (35-9.5)^2(0.3)}$$
$$= 19.37$$

Covariance for Investment Returns

		Investment	
P(ΔX,ΔY)	Economic condition	Bond Fund X	Aggressive Fund Y
0.2	Recession	+ $ 7	- $20
0.5	Stable Economy	+ 4	+ 6
0.3	Expanding Economy	+ 2	+ 35

$$\sigma_{\Delta X \Delta Y} = \text{Cov}(\Delta X, \Delta Y) = (7-4)(-20-9.5)(.2) + (4-4)(6-9.5)(.5) + (2-4)(35-9.5)(.3) = -33$$

Portfolio Example

Investment X: $E(\Delta X) = 4$ $\sigma_{\Delta X} = 1.73$

Investment Y: $E(\Delta Y) = 9.5$ $\sigma_{\Delta Y} = 19.32$

$\sigma_{\Delta X \Delta Y} = -33$

Suppose 40% of the portfolio (W) is in Investment X and 60% is in Investment Y:

$$E(\Delta W) = .4(4) + (.6)(9.5) = 7.3$$

$$\sqrt{\text{Var}(\Delta W)} = \sqrt{(.4)^2(1.73)^2 + (.6)^2(19.32)^2 + 2(.4)(.6)(-33)}$$
$$= 10.91$$

The portfolio return and portfolio variability are between the values for investments X and Y considered individually

Interpreting the Results for Investment Returns

- The aggressive fund has a higher expected return, but much more risk

$$E(\Delta Y) = 9.5 \ > E(\Delta X) = 4$$

but

$$\sigma_{\Delta Y} = 19.32 \ > \sigma_{\Delta X} = 1.73$$

- The Covariance of -33 indicates that the two investments are negatively related and will vary in the opposite direction