Statistics for Business and Economics

Chapter 7

Estimation: Single Population

Confidence Intervals

Contents of this chapter:

- Confidence Intervals for the Population Mean, μ
  - when Population Variance σ^2 is Known
  - when Population Variance σ^2 is Unknown
- Confidence Intervals for the Population Proportion, $\hat{p}$ (large samples)
- Confidence interval estimates for the variance of a normal population

7.1

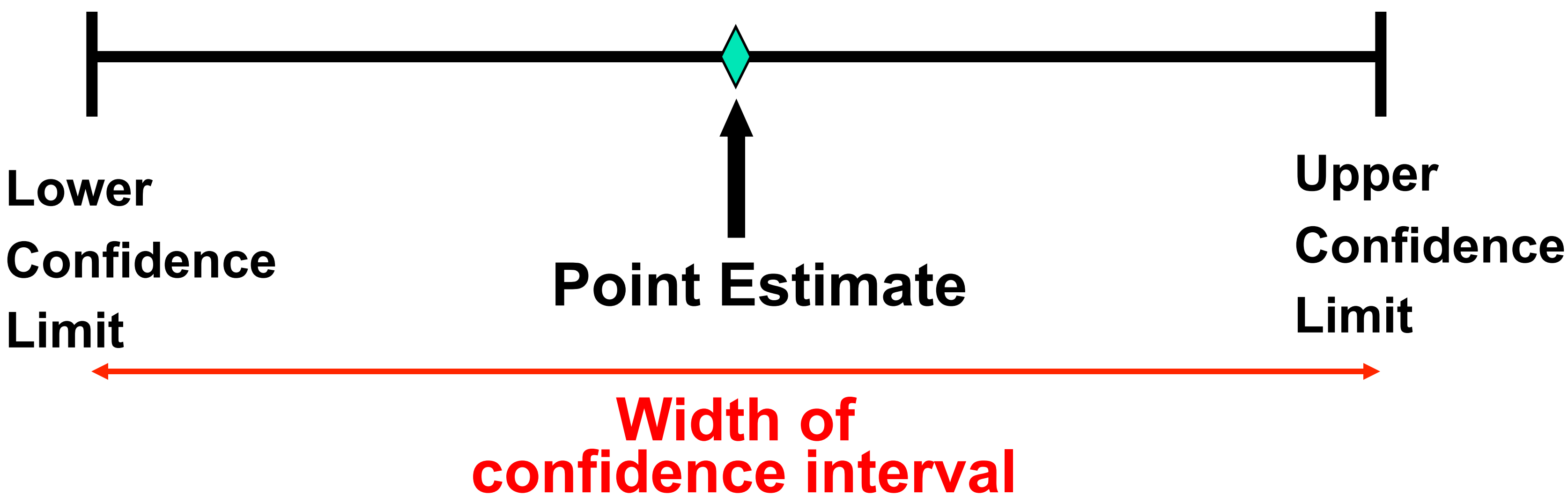
Definitions

- An estimator of a population parameter is
  - a random variable that depends on sample information . . .
  - whose value provides an approximation to this unknown parameter

- A specific value of that random variable is called an estimate

Point and Interval Estimates

- A point estimate is a single number,
- a confidence interval provides additional information about variability



Point Estimates

We can estimate a Population Parameter …		with a Sample Statistic (a Point Estimate)
Mean	μ	$\bar{x}$
Variance	σ^2	s^2

Unbiasedness

- A point estimator $\hat{\theta}$ is said to be an unbiased estimator of the parameter θ if the expected value, or mean, of the sampling distribution of $\hat{\theta}$ is θ,

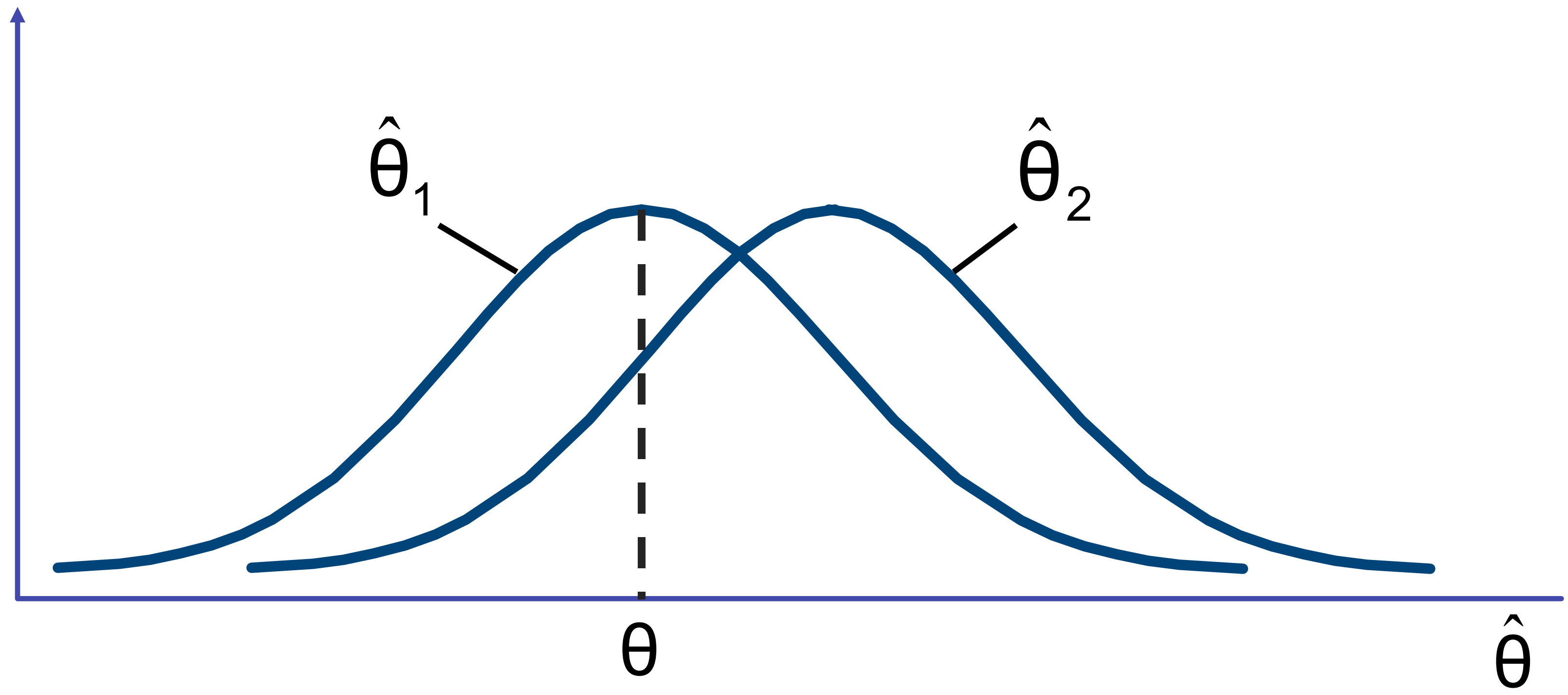
$$\boxed{E(\hat{\theta}) = \theta}$$

- Examples:
  - The sample mean $\bar{x}$ is an unbiased estimator of μ
  - The sample variance s^2 is an unbiased estimator of σ^2
  - The sample proportion $\hat{p}$ is an unbiased estimator of P

Unbiasedness

(continued)

- $\hat{\theta}_1$ is an unbiased estimator, $\hat{\theta}_2$ is biased:



Bias

- Let $\hat{\theta}$ be an estimator of θ

- The bias in $\hat{\theta}$ is defined as the difference between its mean and θ

$$\boxed{\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta}$$

- The bias of an unbiased estimator is 0

Most Efficient Estimator

- Suppose there are several unbiased estimators of θ
- The most efficient estimator or the minimum variance unbiased estimator of θ is the unbiased estimator with the smallest variance
- Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be two unbiased estimators of θ, based on the same number of sample observations. Then,
  - $\hat{\theta}_1$ is said to be more efficient than $\hat{\theta}_2$ if $\boxed{\mathrm{Var}(\hat{\theta}_1) < \mathrm{Var}(\hat{\theta}_2)}$
  - The relative efficiency of $\hat{\theta}_1$ with respect to $\hat{\theta}_2$ is the ratio of their variances:

$$\boxed{\text{Relative Efficiency} = \frac{\mathrm{Var}(\hat{\theta}_2)}{\mathrm{Var}(\hat{\theta}_1)}}$$

7.2

Confidence Intervals

- How much uncertainty is associated with a point estimate of a population parameter?

- An interval estimate provides more information about a population characteristic than does a point estimate

- Such interval estimates are called confidence intervals

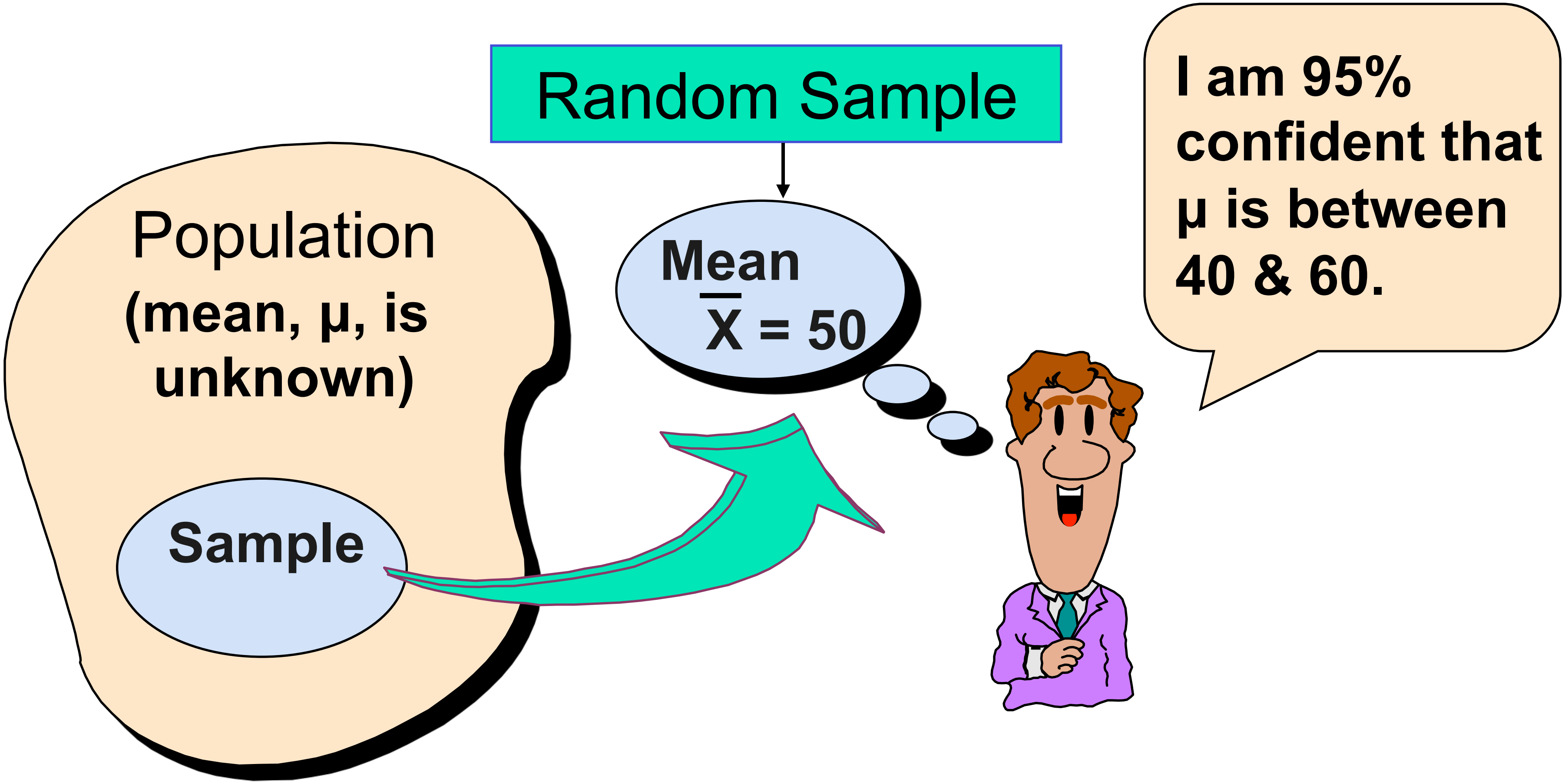
Confidence Interval Estimate

- An interval gives a range of values:
  - Takes into consideration variation in sample statistics from sample to sample
  - Based on observation from 1 sample
  - Stated in terms of level of confidence
    - Can never be 100% confident

Confidence Interval and Confidence Level

- If $P(a < \theta < b) = 1 - \alpha$ then the interval from a to b is called a $100(1 - \alpha)\%$ confidence interval of θ.
- The quantity $(1 - \alpha)$ is called the confidence level of the interval (α between 0 and 1)
  - In repeated samples of the population, the true value of the parameter θ would be contained in $100(1 - \alpha)\%$ of intervals calculated this way.
  - The confidence interval is written as LCL $< \theta <$ UCL with $100(1 - \alpha)\%$ confidence

Estimation Process



Confidence Level, $(1-\alpha)$

(continued)

- Suppose confidence level = 95%
- $(1 - \alpha) = 0.95$
- From repeated samples, 95% of all the confidence intervals that can be constructed will contain the unknown true parameter
- A specific interval either will contain or will not contain the true parameter
  - No probability involved in a specific interval

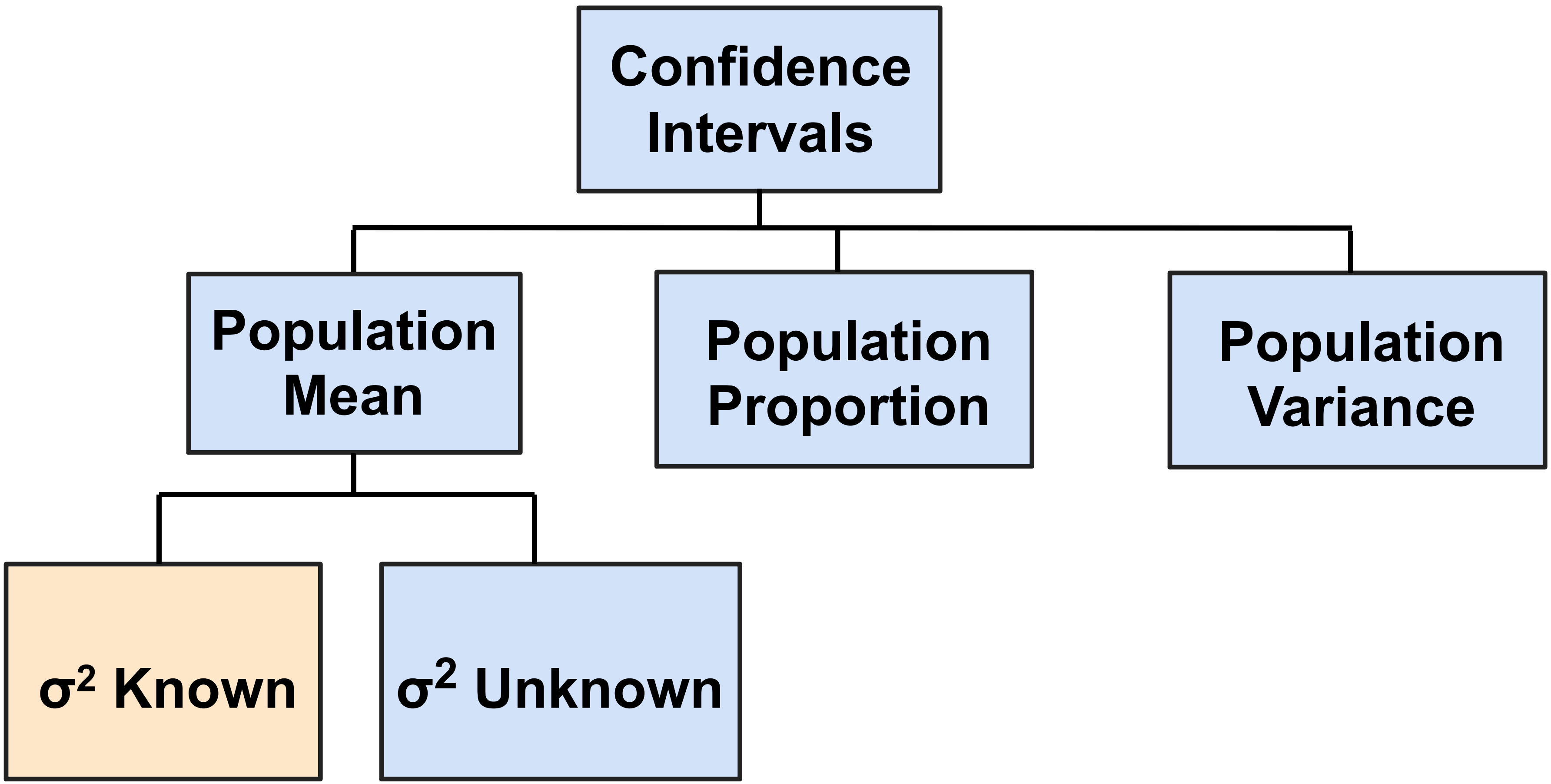
General Formula

- The general formula for all confidence intervals is:

Point Estimate ± (Reliability Factor)(Standard Error)

- The value of the reliability factor depends on the desired level of confidence

Confidence Intervals



7.2

Confidence Interval for μ (σ^2 Known)

- Assumptions
  - Population variance σ^2 is known
  - Population is normally distributed
  - If population is not normal, use large sample
- Confidence interval estimate:

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

(where $z_{\alpha/2}$ is the normal distribution value for a probability of $\alpha/2$ in each tail)

Margin of Error

- The confidence interval,

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

- Can also be written as $\boxed{\bar{x} \pm ME}$
  where ME is called the margin of error

$$\boxed{ME = z_{\alpha/2}\frac{\sigma}{\sqrt{n}}}$$

- The interval width, w, is equal to twice the margin of error

Reducing the Margin of Error

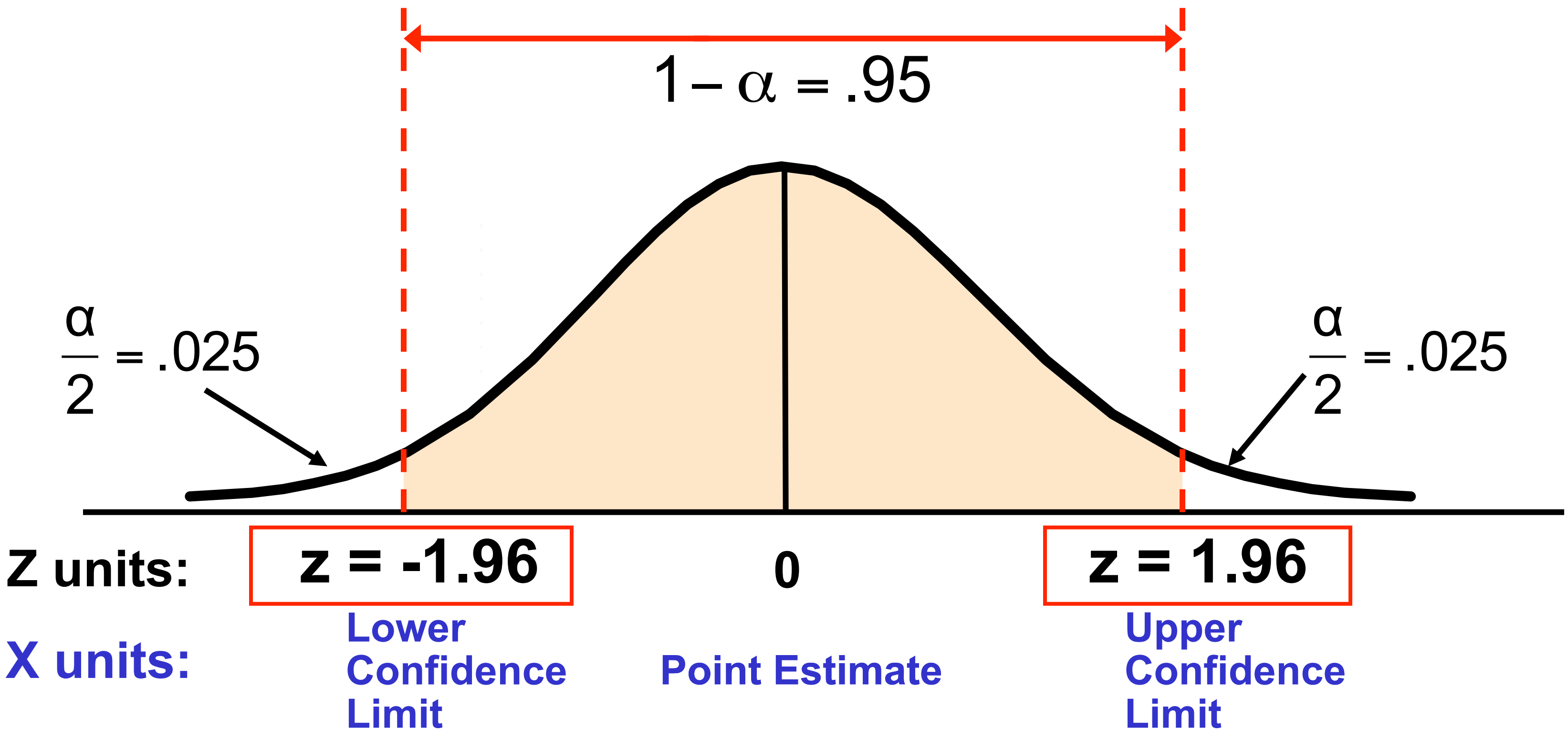
$$ME = z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

The margin of error can be reduced if

- the population standard deviation can be reduced ($\sigma\downarrow$)
- The sample size is increased ($n\uparrow$)
- The confidence level is decreased, $(1 - \alpha) \downarrow$

Finding the Reliability Factor, $z_{\alpha/2}$

- Consider a 95% confidence interval:



- Find $z_{.025} = \pm 1.96$ from the standard normal distribution table

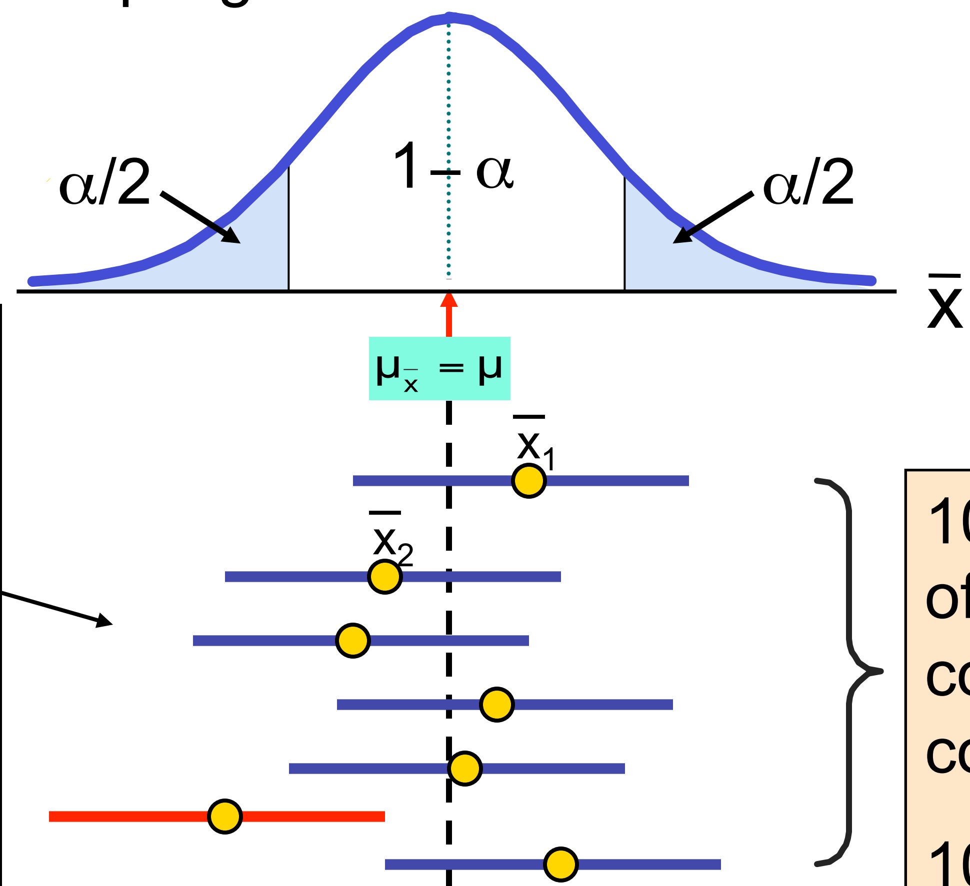
Common Levels of Confidence

- Commonly used confidence levels are 90%, 95%, and 99%

Confidence Level	***Confidence Coefficient,*** $1-\alpha$	$Z_{\alpha/2}$ ***value***
80%	**.80**	**1.28**
90%	**.90**	**1.645**
95%	**.95**	**1.96**
98%	**.98**	**2.33**
99%	**.99**	**2.58**
99.8%	**.998**	**3.08**
99.9%	**.999**	**3.27**

Intervals and Level of Confidence

Sampling Distribution of the Mean



Confidence Intervals

Intervals extend from

$$LCL = \bar{x} - z\frac{\sigma}{\sqrt{n}}$$

to

$$UCL = \bar{x} + z\frac{\sigma}{\sqrt{n}}$$

$100(1-\alpha)\%$ of intervals constructed contain μ;

$100(\alpha)\%$ do not.

Example

- A sample of 27 light bulb from a large normal population has a mean life length of 1478 hours. We know that the population standard deviation is 36 hours.

- Determine a 95% confidence interval for the true mean length of life in the population.

Example

(continued)

- Solution:

$$\bar{x} \pm z \frac{\sigma}{\sqrt{n}}$$

$$= 1478 \pm 1.96\,(36/\sqrt{27})$$

$$= 1478 \pm 13.58$$

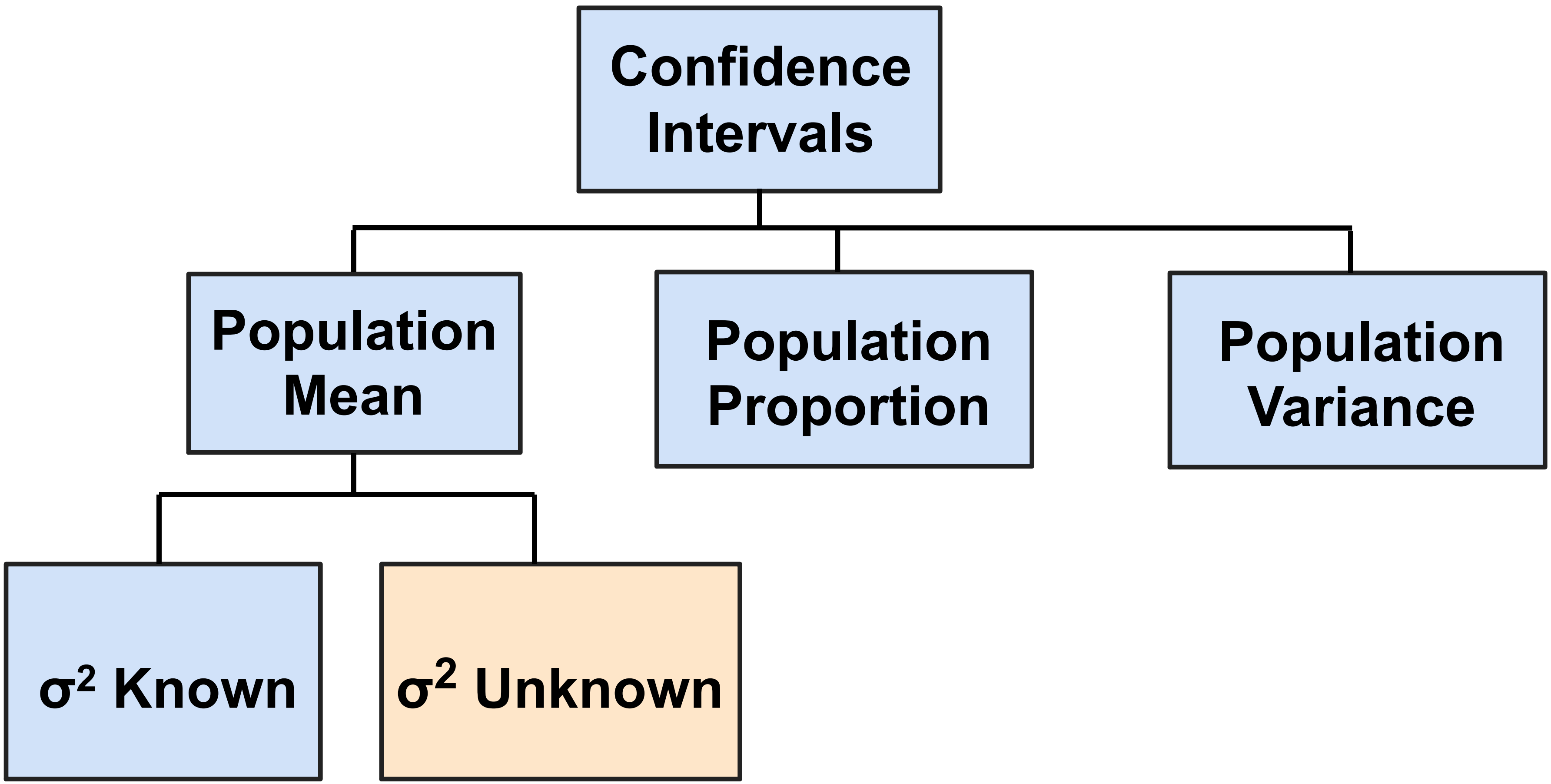
$$1464.42 < \mu < 1491.58$$

Interpretation

- We are 95% confident that the true mean life time is between 1464.42 and 1491.58
- Although the true mean may or may not be in this interval, 95% of intervals formed in this manner will contain the true mean

7.3

Confidence Intervals



Student's t Distribution

- Consider a random sample of n observations
  - with mean $\bar{x}$ and standard deviation s
  - from a normally distributed population with mean μ

- Then the variable

$$\boxed{t = \frac{\bar{x} - \mu}{s/\sqrt{n}}}$$

follows the Student's t distribution with (n - 1) degrees of freedom

Confidence Interval for μ (σ^2 Unknown)

- If the population standard deviation σ is unknown, we can substitute the sample standard deviation, s
- This introduces extra uncertainty, since s is variable from sample to sample
- So we use the t distribution instead of the normal distribution

Confidence Interval for μ (σ Unknown)

(continued)

- Assumptions
  - Population standard deviation is unknown
  - Population is normally distributed
  - If population is not normal, use large sample
- Use Student's t Distribution
- Confidence Interval Estimate:

$$\bar{x} - t_{n-1,\alpha/2}\frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{n-1,\alpha/2}\frac{s}{\sqrt{n}}$$

where $t_{n-1,\alpha/2}$ is the critical value of the t distribution with n-1 d.f. and an area of $\alpha/2$ in each tail: $P(t_{n-1} > t_{n-1,\alpha/2}) = \alpha/2$

Margin of Error

- The confidence interval,

$$\bar{x} - t_{n-1,\alpha/2}\frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{n-1,\alpha/2}\frac{s}{\sqrt{n}}$$

- Can also be written as $\bar{x} \pm ME$

where ME is called the margin of error:

$$ME = t_{n-1,\alpha/2}\frac{\sigma}{\sqrt{n}}$$

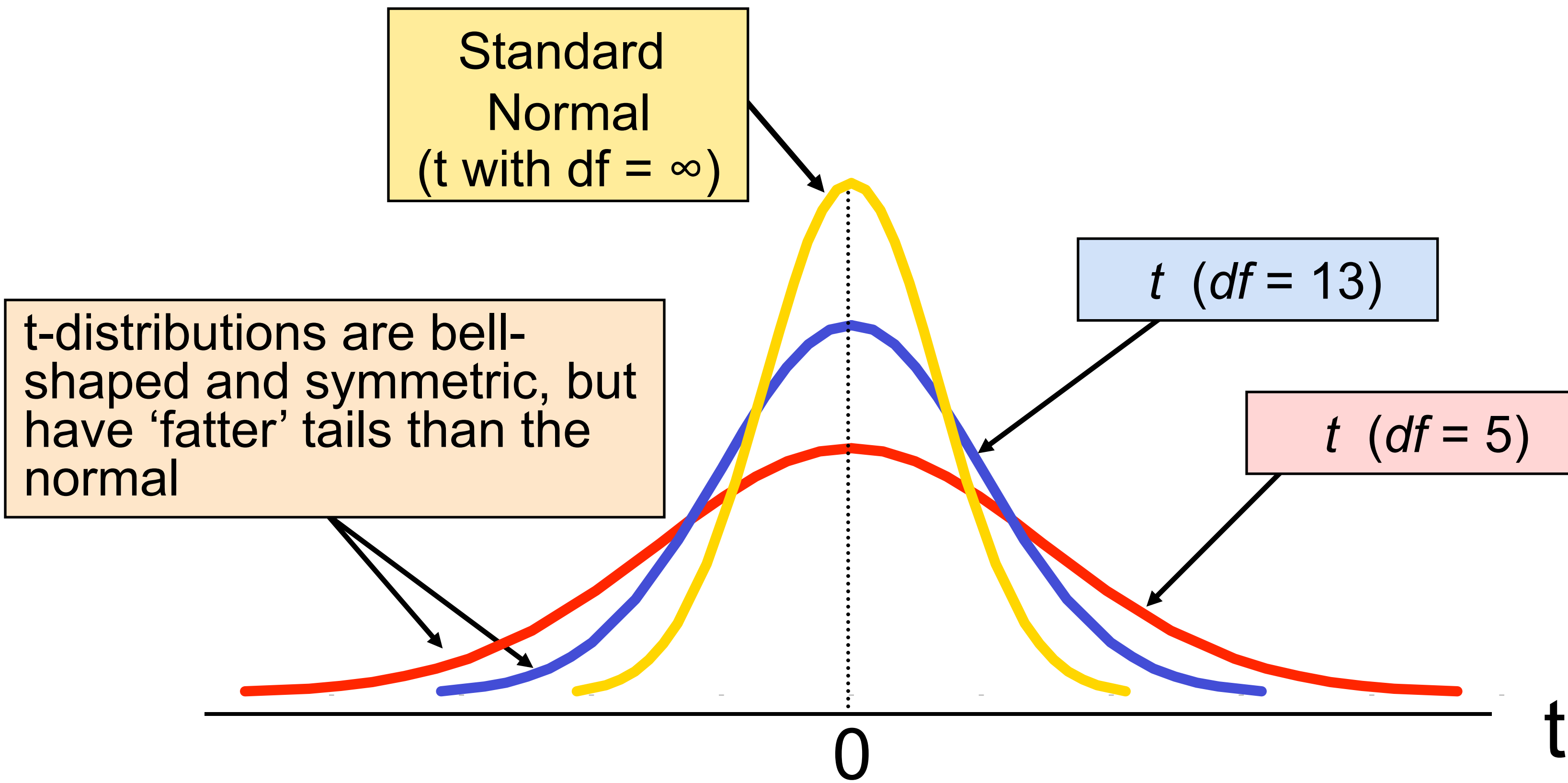
Student's t Distribution

- The t is a family of distributions
- The t value depends on degrees of freedom (d.f.)
  - Number of observations that are free to vary after sample mean has been calculated

$$\boxed{d.f. = n - 1}$$

Student's t Distribution

Note: $t \longrightarrow Z$ as n increases

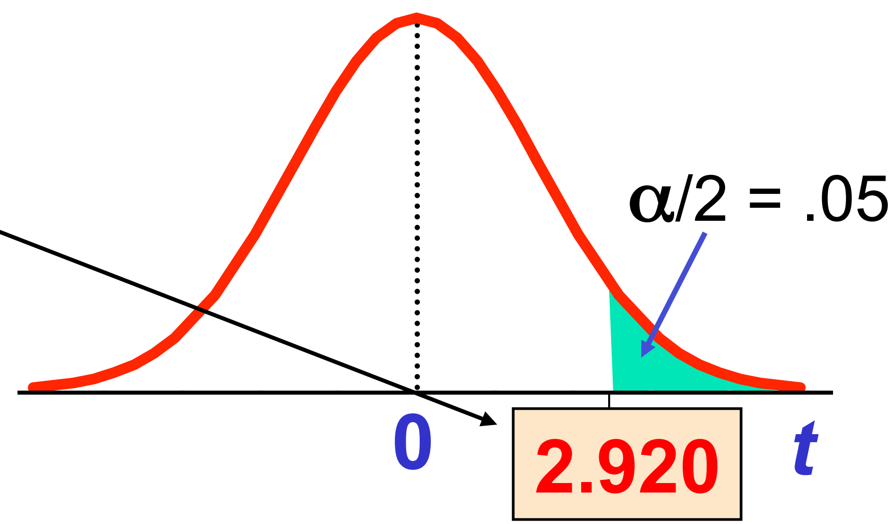


Student's t Table

df	Upper Tail Area		
	.10	**.05**	.025
1	3.078	6.314	12.706
2	1.886	**2.920**	4.303
3	1.638	2.353	3.182

The body of the table contains t values, not probabilities

Let: n = 3
df = n - 1 = 2
α = .10
α/2 =.05



t distribution values

With comparison to the Z value

Confidence Level	t (10 d.f.)	t (20 d.f.)	t (30 d.f.)	Z
.80	1.372	1.325	1.310	1.282
.90	1.812	1.725	1.697	1.645
.95	2.228	2.086	2.042	1.960
.99	3.169	2.845	2.750	2.576

Note: $t \longrightarrow Z$ as n increases

Example

A random sample of n = 25 has $\bar{x}$ = 50 and s = 8. Form a 95% confidence interval for μ

- d.f. = n – 1 = 24, so $t_{n-1,\alpha/2} = t_{24,.025} = 2.0639$

The confidence interval is

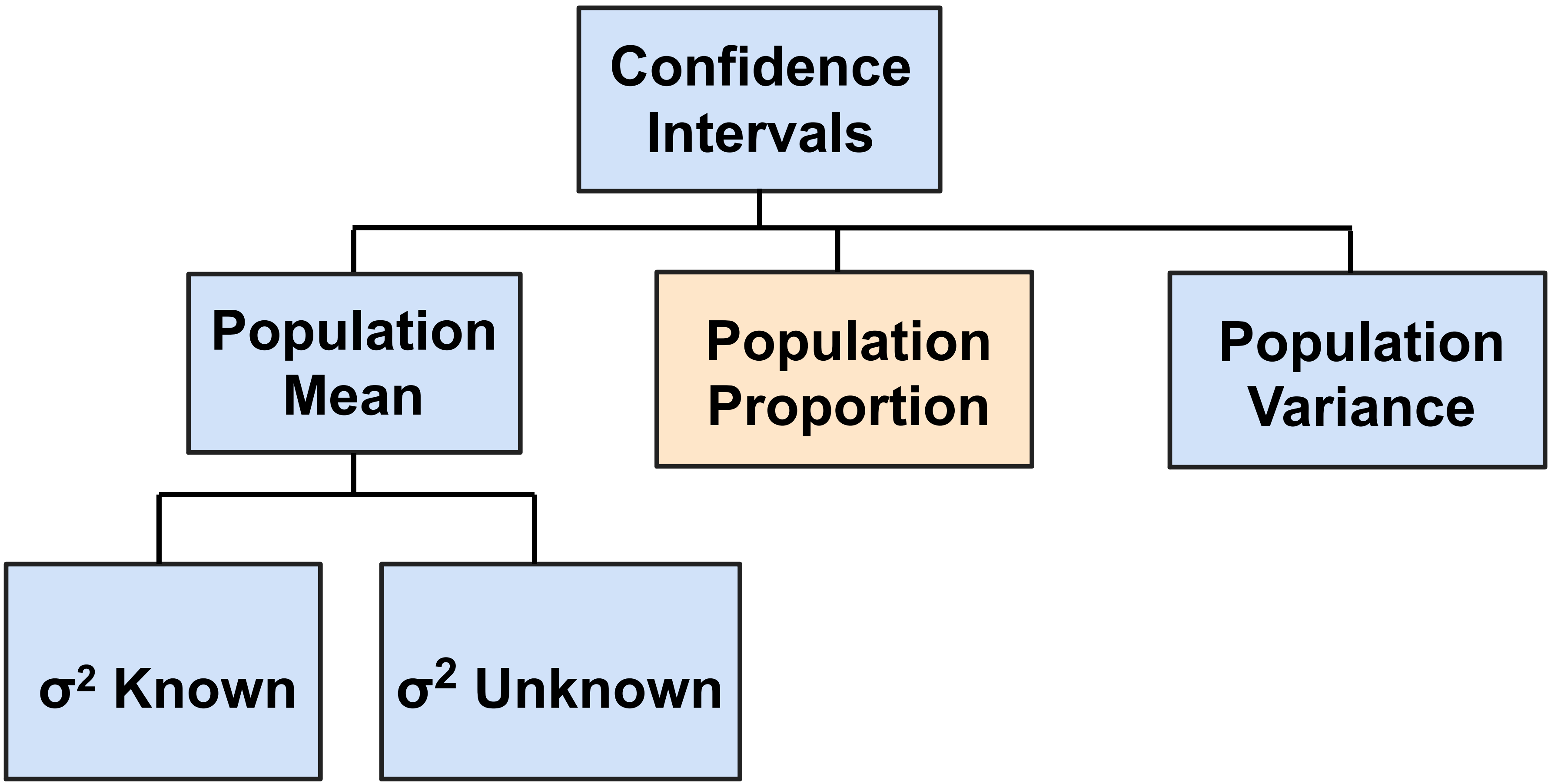
$$\bar{x} - t_{n\text{-}1,\alpha/2}\frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{n\text{-}1,\alpha/2}\frac{s}{\sqrt{n}}$$

$$50 - (2.0639)\frac{8}{\sqrt{25}} < \mu < 50 + (2.0639)\frac{8}{\sqrt{25}}$$

$$\boxed{46.698 < \mu < 53.302}$$

7.4

Confidence Intervals



Confidence Intervals for the Population Proportion

- An interval estimate for the population proportion (p) can be calculated by adding an allowance for uncertainty to the sample proportion ($\hat{p}$)

Confidence Intervals for the Population Proportion, p

(continued)

- Recall that the distribution of the sample proportion is approximately normal if the sample size is large, with standard deviation

$$\sigma_p = \sqrt{\frac{p(1-p)}{n}}$$

- We will estimate this with sample data:

$$\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Confidence Interval Endpoints

- Upper and lower confidence limits for the population proportion are calculated with the formula

$$\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \;<\; p \;<\; \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

- where
  - $z_{\alpha/2}$ is the standard normal value for the level of confidence desired
  - $\hat{p}$ is the sample proportion
  - n is the sample size

Example

- A random sample of 100 people shows that 25 are left-handed.
- Form a 95% confidence interval for the true proportion of left-handers

Example

(continued)

- A random sample of 100 people shows that 25 are left-handed. Form a 95% confidence interval for the true proportion of left-handers.

$$\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \; < \; p \; < \; \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$\frac{25}{100} - 1.96\sqrt{\frac{.25(.75)}{100}} \; < \; p \; < \; \frac{25}{100} + 1.96\sqrt{\frac{.25(.75)}{100}}$$

$$\boxed{0.1651 \; < \; p \; < \; 0.3349}$$

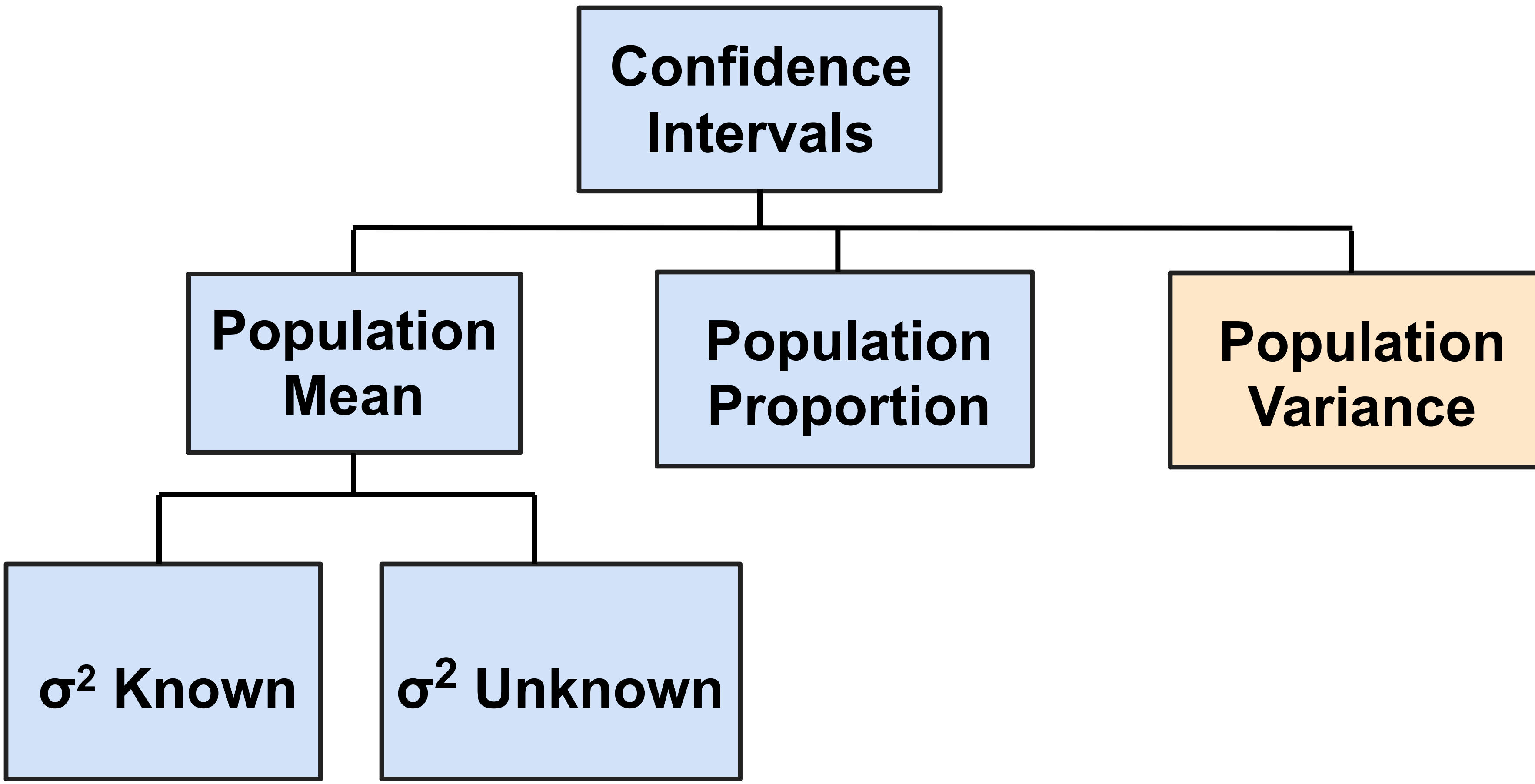
Interpretation

- We are 95% confident that the true percentage of left-handers in the population is between

  16.51% and 33.49%.

- Although the interval from 0.1651 to 0.3349 may or may not contain the true proportion, 95% of intervals formed from samples of size 100 in this manner will contain the true proportion.

Confidence Intervals



Confidence Intervals for the Population Variance

- Goal: Form a confidence interval for the population variance, σ^2

  - The confidence interval is based on the sample variance, s^2

  - Assumed: the population is normally distributed

Confidence Intervals for the Population Variance

(continued)

The random variable

$$\boxed{\chi^2_{n-1} = \frac{(n-1)s^2}{\sigma^2}}$$

follows a chi-square distribution with (n – 1) degrees of freedom

Where the chi-square value $\chi^2_{n-1,\alpha}$ denotes the number for which

$$\boxed{P(\chi^2_{n-1} > \chi^2_{n-1,\alpha}) = \alpha}$$

Confidence Intervals for the Population Variance

(continued)

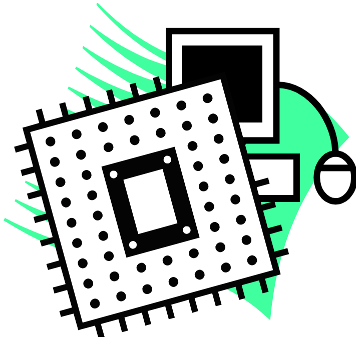
The (1 - α)% confidence interval for the population variance is

$$\boxed{\frac{(n-1)s^2}{\chi^2_{n-1,\,\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{n-1,\,1\text{-}\alpha/2}}}$$

Example

You are testing the speed of a batch of computer processors. You collect the following data (in Mhz):

Sample size	**17**
Sample mean	**3004**
Sample std dev	**74**



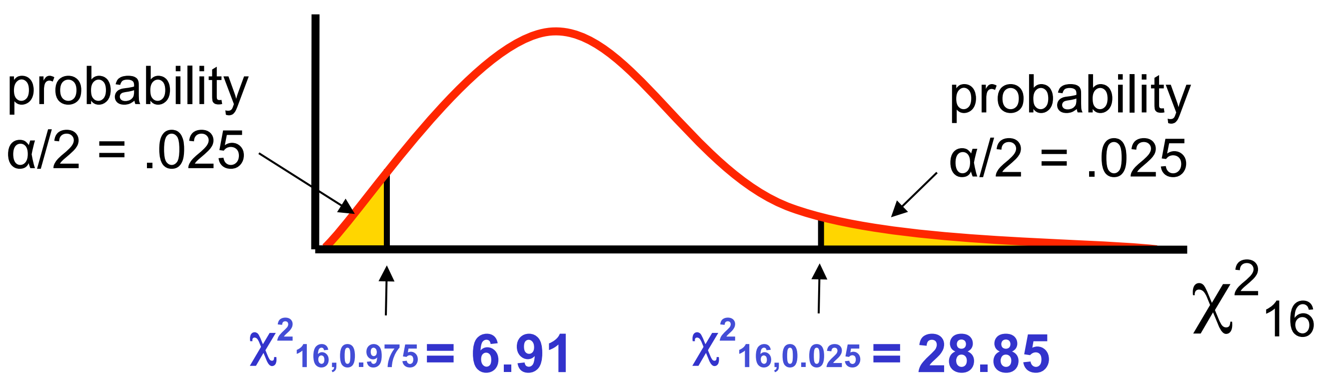
Assume the population is normal.
Determine the 95% confidence interval for σ_x^2

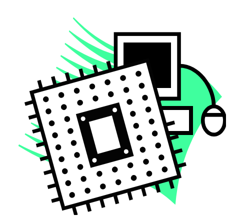
Finding the Chi-square Values

- n = 17 so the chi-square distribution has (n – 1) = 16 degrees of freedom
- α = 0.05, so use the the chi-square values with area 0.025 in each tail:

$$\chi^2_{n-1,\,1\text{-}\alpha/2} = \chi^2_{16,\,0.975} = 6.91$$

$$\chi^2_{n-1,\,\alpha/2} = \chi^2_{16,\,0.025} = 28.85$$





Calculating the Confidence Limits

- The 95% confidence interval is

$$\frac{(n-1)s^2}{\chi^2_{n-1\,,\,\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{n-1\,,\,1-\alpha/2}}$$

$$\frac{(17-1)(74)^2}{28.85} < \sigma^2 < \frac{(17-1)(74)^2}{6.91}$$

$$\boxed{3037 < \sigma^2 < 12683}$$

Converting to standard deviation, we are 95% confident that the population standard deviation of CPU speed is between 55.1 and 112.6 Mhz