Statistics for Business and Economics

Chapter 9

Hypothesis Testing: Single Population

9.1

What is a Hypothesis?

- A hypothesis is a claim (assumption) about a population parameter:

  - population mean

Example: The mean monthly cell phone bill of this city is $\mu = \$42$

  - population proportion

Example: The proportion of adults in this city with cell phones is $p = .68$

The Null Hypothesis, H_0

- States the assumption (numerical) to be tested

Example: The average number of TV sets in household is equal to three ($H_0: \mu = 3$)

- Is always about a population parameter, not about a sample statistic

$$H_0: \mu = 3 \qquad \cancel{H_0: \bar{X} = 3}$$

The Null Hypothesis, H_0

(continued)

- Begin with the assumption that the null hypothesis is true
  - Similar to the notion of innocent until proven guilty

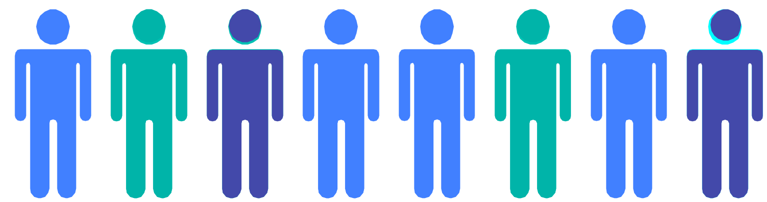
- Refers to the status quo
- Always contains "=" , "≤" or "≥" sign
- May or may not be rejected

The Alternative Hypothesis, H_1

- Is the opposite of the null hypothesis
  - e.g., The average number of TV sets in U.S. homes is not equal to 3 ($H_1: \mu \neq 3$)
- Challenges the status quo
- Never contains the "=" , "≤" or "≥" sign
- May or may not be supported
- Is generally the hypothesis that the researcher is trying to support

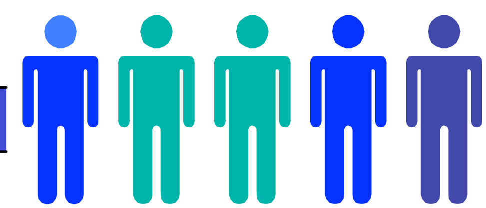
Hypothesis Testing Process

Claim: the population mean age is 50. (Null Hypothesis: H_0: $\mu = 50$)



Population

Now select a random sample



Sample

Suppose the sample mean age is 20: $\bar{X} = 20$

Is $\bar{X} = 20$ likely if $\mu = 50$?

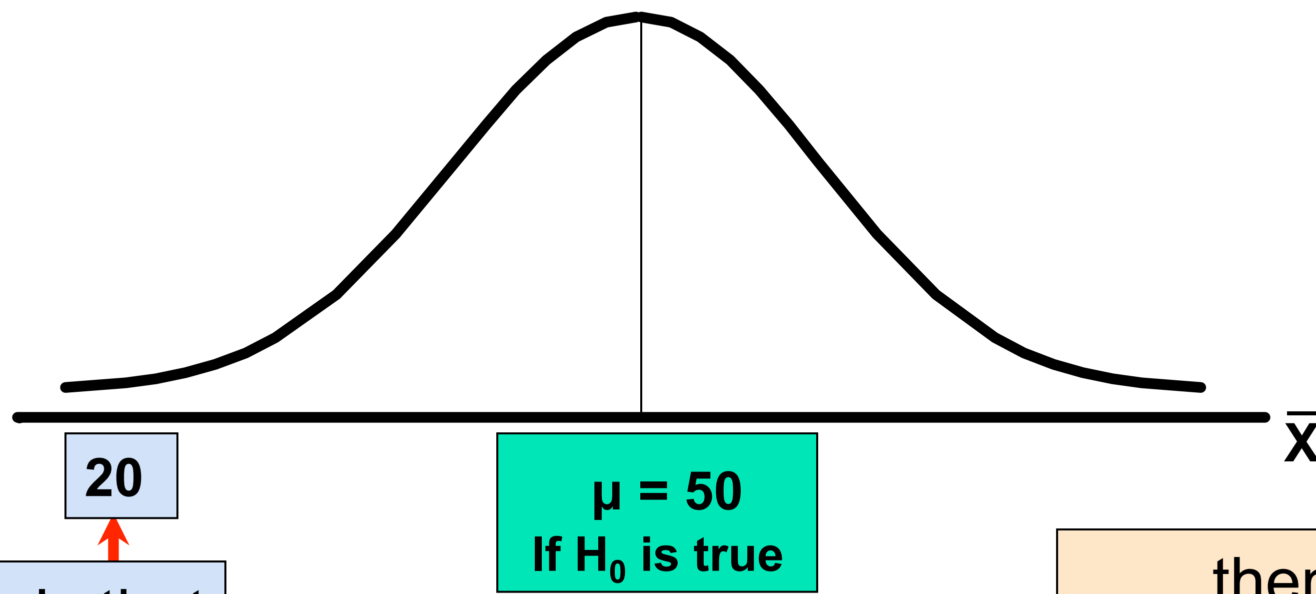
If not likely,

REJECT

Null Hypothesis

Reason for Rejecting H_0

Sampling Distribution of $\bar{X}$



If it is unlikely that we would get a sample mean of this value ...

... if in fact this were the population mean…

... then we reject the null hypothesis that μ = 50.

Level of Significance, α

- **Defines the unlikely values of the sample statistic if the null hypothesis is true**
  - Defines rejection region of the sampling distribution
- Is designated by α , (level of significance)
  - Typical values are .01, .05, or .10
- Is selected by the researcher at the beginning
- Provides the critical value(s) of the test

Level of Significance and the Rejection Region

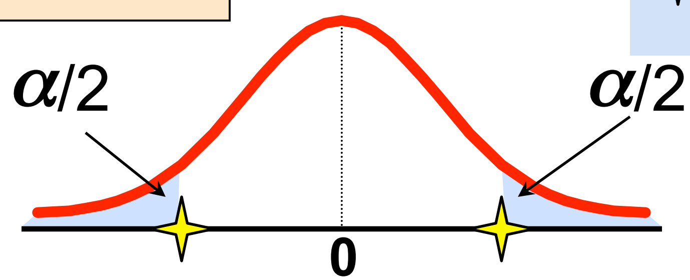
Level of significance = α

Represents critical value

Rejection region is shaded

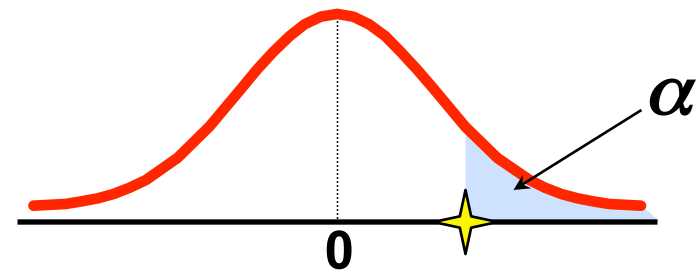
H_0: $\mu = 3$
H_1: $\mu \neq 3$

Two-tail test



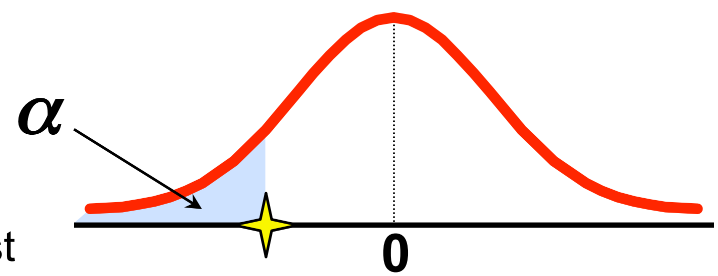
H_0: $\mu \leq 3$
H_1: $\mu > 3$

Upper-tail test



H_0: $\mu \geq 3$
H_1: $\mu < 3$

Lower-tail test



Errors in Making Decisions

- **Type I Error**
  - Reject a true null hypothesis
  - Considered a serious type of error

The probability of Type I Error is α

- Called level of significance of the test
- Set by researcher in advance

Errors in Making Decisions

(continued)

- **Type II Error**
  - Fail to reject a false null hypothesis

The probability of Type II Error is β

Outcomes and Probabilities

Possible Hypothesis Test Outcomes

	Actual Situation	
Decision	H_0 True	H_0 False
Do Not Reject H_0	**No Error** **$(1 - \alpha)$**	**Type II Error** **(β)**
Reject H_0	**Type I Error** **(α)**	**No Error** **$(1 - \beta)$**

Key:
Outcome
(Probability)

Type I & II Error Relationship

- Type I and Type II errors can not happen at the same time
  - Type I error can only occur if H_0 is true
  - Type II error can only occur if H_0 is false

If Type I error probability (α) ↑, then

Type II error probability (β) ↓

Factors Affecting Type II Error

- All else equal,
  - $\beta \uparrow$ when the difference between hypothesized parameter and its true value $\downarrow$
  - $\beta \uparrow$ when $\alpha \downarrow$
  - $\beta \uparrow$ when $\sigma \uparrow$
  - $\beta \uparrow$ when $n \downarrow$

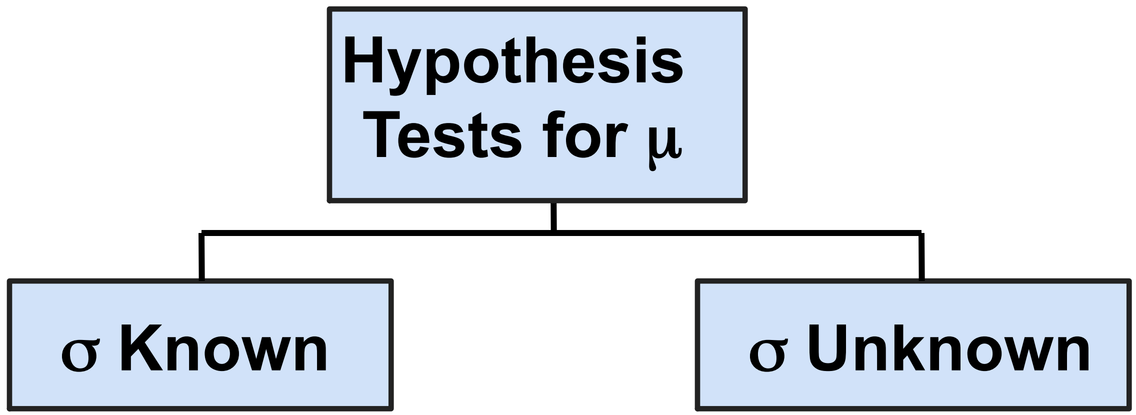
Power of the Test

- The power of a test is the probability of rejecting a null hypothesis that is false

- i.e., Power = P(Reject H_0 | H_1 is true)

  - Power of the test increases as the sample size increases

Hypothesis Tests for the Mean



Test of Hypothesis for the Mean (σ Known)

9.2

- Convert sample result ($\bar{x}$) to a z value

Hypothesis Tests for μ

σ Known | **σ Unknown**

Consider the test

$$H_0: \mu = \mu_0$$
$$H_1: \mu > \mu_0$$

(Assume the population is normal)

The decision rule is:

$$\text{Reject } H_0 \text{ if } z = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \geq z_\alpha$$

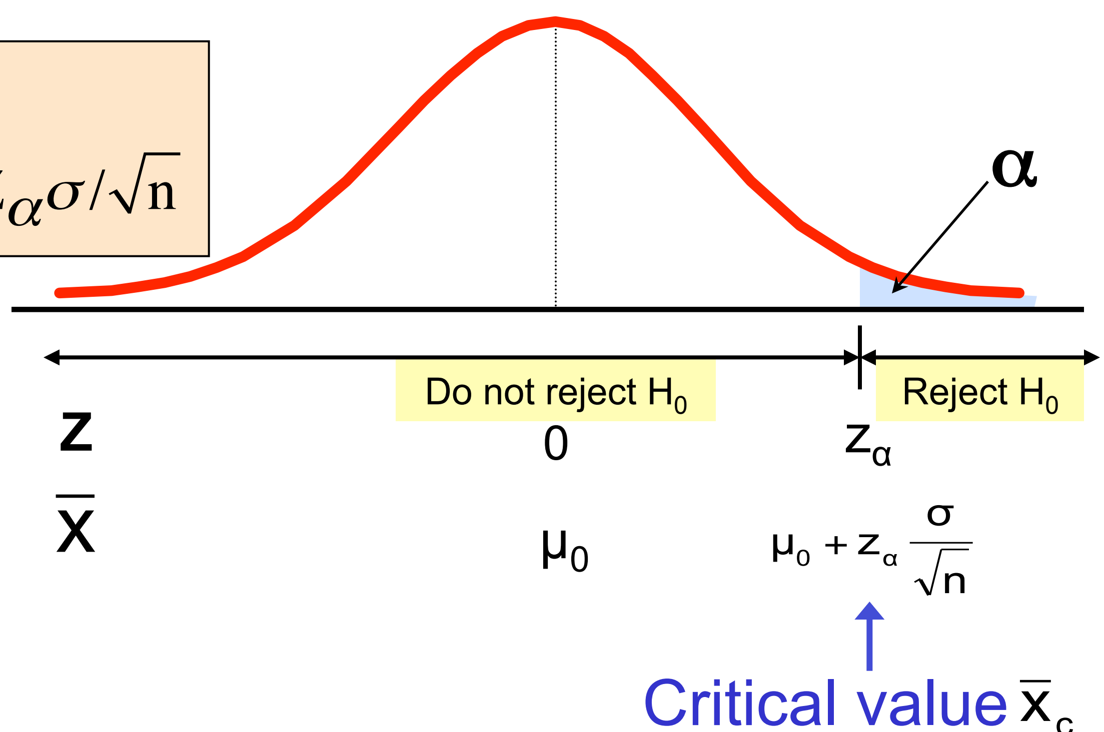
Decision Rule

$H_0: \mu = \mu_0$
$H_1: \mu > \mu_0$

$$\text{Reject } H_0 \text{ if } z = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \geq z_\alpha$$

Alternate rule:

$$\text{Reject } H_0 \text{ if } \bar{x} \geq \mu_0 + Z_\alpha \sigma / \sqrt{n}$$



p-Value Approach to Testing

- p-value: Probability of obtaining a test statistic more extreme (≤ or ≥) than the observed sample value given H_0 is true
  - Also called observed level of significance
  - Smallest value of α for which H_0 can be rejected

p-Value Approach to Testing

(continued)

- Convert sample result (e.g., $\bar{x}$) to test statistic (e.g., z statistic)
- Obtain the p-value
  - For an upper tail test:

$$\text{p-value} = P\left(z \geq \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\text{, given that } H_0 \text{ is true}\right)$$
$$= P\left(z \geq \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \mid \mu = \mu_0\right)$$

- Decision rule: compare the p-value to α

  - If p-value $\leq \alpha$, reject H_0
  - If p-value $> \alpha$, do not reject H_0

Example: Upper-Tail Z Test for Mean (σ Known)

A phone industry manager thinks that customer monthly cell phone bill have increased, and now average over \$52 per month. The company wishes to test this claim. (Assume $\sigma = 10$ is known)

Form hypothesis test:

H_0: $\mu \le 52$ the average is not over \$52 per month

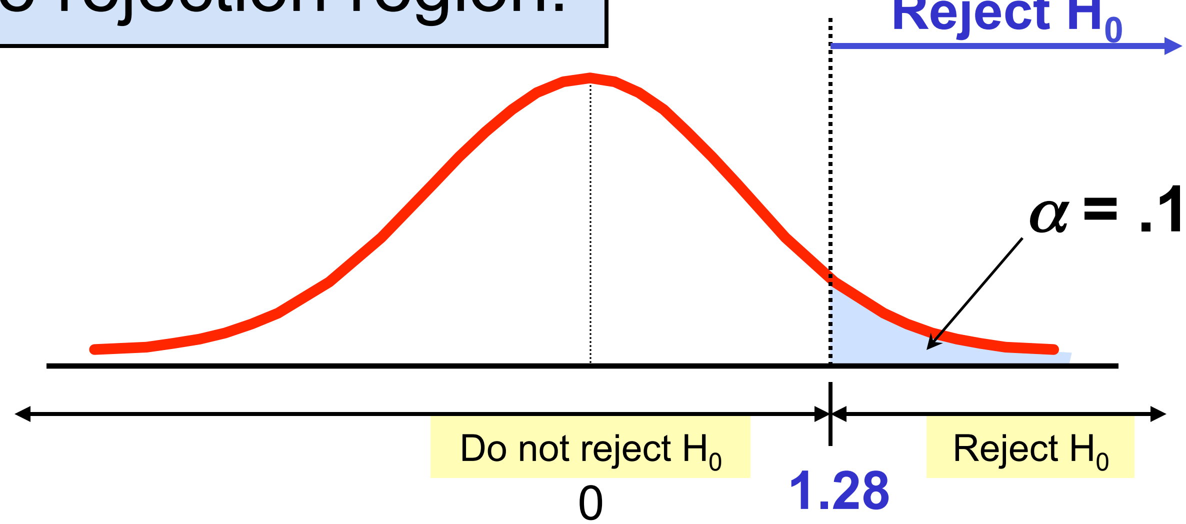
H_1: $\mu > 52$ the average is greater than \$52 per month (i.e., sufficient evidence exists to support the manager's claim)

Example: Find Rejection Region

(continued)

- Suppose that α = .10 is chosen for this test

Find the rejection region:



$$\text{Reject } H_0 \text{ if } z = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \geq 1.28$$

Example: Sample Results

(continued)

Obtain sample and compute the test statistic

Suppose a sample is taken with the following results: $n = 64$, $\bar{x} = 53.1$ ($\sigma = 10$ was assumed known)

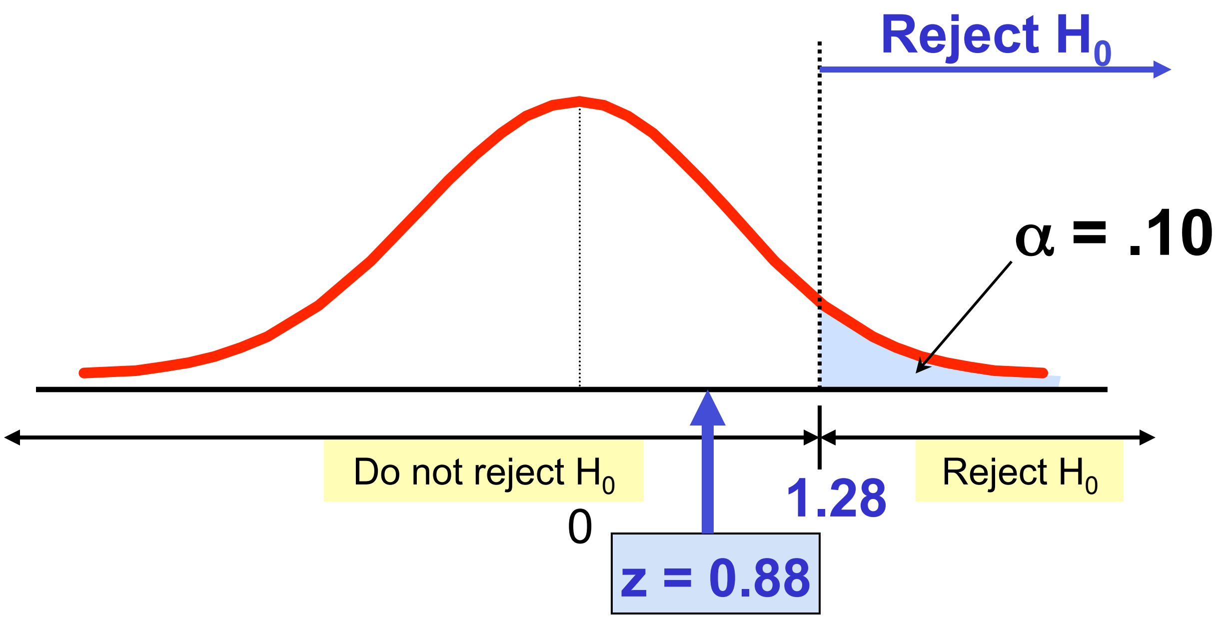
- Using the sample results,

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{53.1 - 52}{\frac{10}{\sqrt{64}}} = 0.88$$

Example: Decision

(continued)

Reach a decision and interpret the result:



Do not reject H_0 since z = 0.88 < 1.28

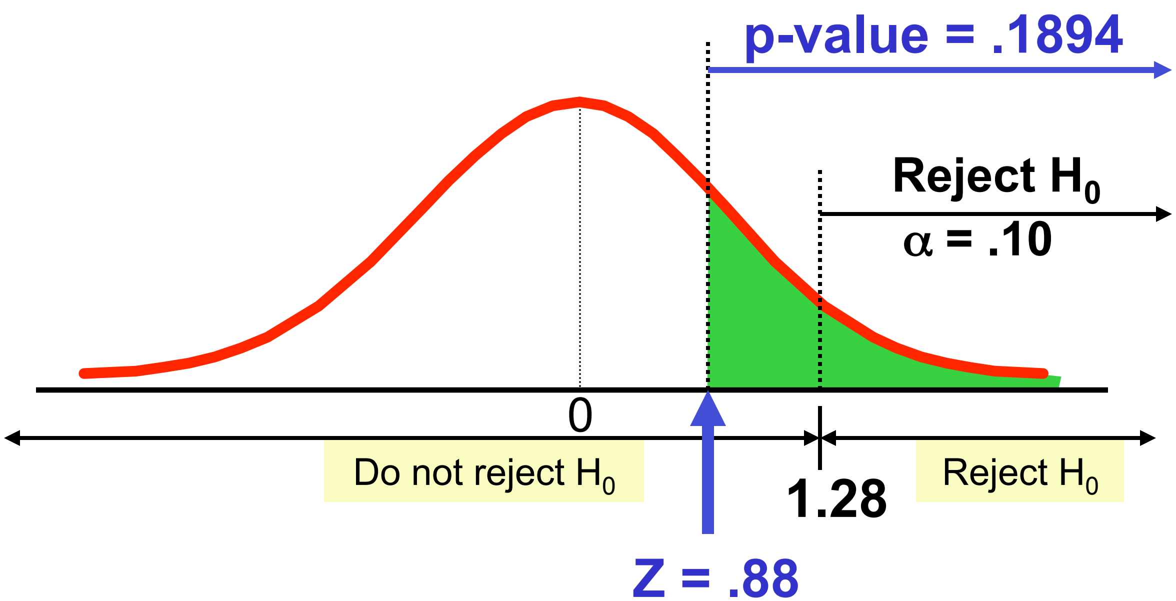
i.e.: there is not sufficient evidence that the mean bill is over $52

Example: p-Value Solution

(continued)

Calculate the p-value and compare to α

(assuming that μ = 52.0)



$$P(\bar{x} \ge 53.1 \mid \mu = 52.0)$$

$$= P\left(z \ge \frac{53.1 - 52.0}{10/\sqrt{64}}\right)$$

$$= P(z \ge 0.88) = 1 - .8106$$

$$= .1894$$

Do not reject H_0 since p-value = .1894 > α = .10

One-Tail Tests

- In many cases, the alternative hypothesis focuses on one particular direction

$H_0: \mu \leq 3$
$H_1: \mu > 3$

→ This is an upper-tail test since the alternative hypothesis is focused on the upper tail above the mean of 3

$H_0: \mu \geq 3$
$H_1: \mu < 3$

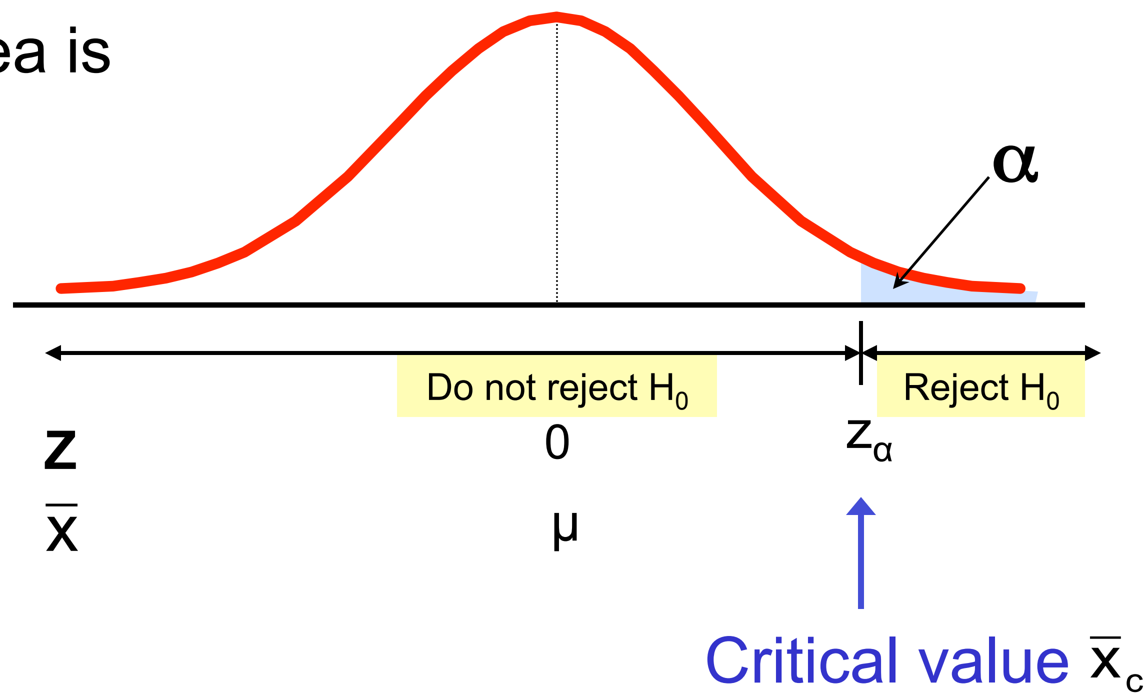
→ This is a lower-tail test since the alternative hypothesis is focused on the lower tail below the mean of 3

Upper-Tail Tests

- There is only one critical value, since the rejection area is in only one tail

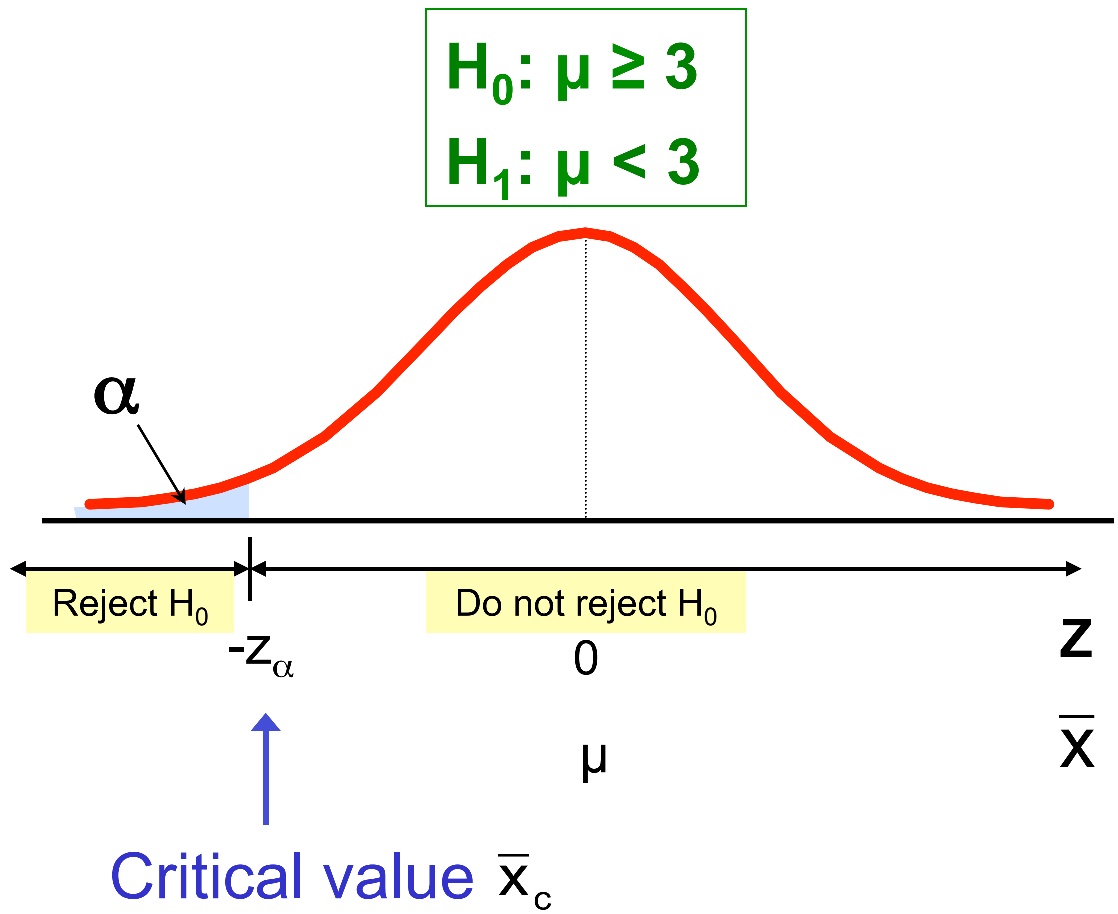
H_0: $\mu \le 3$

H_1: $\mu > 3$



Lower-Tail Tests

- There is only one critical value, since the rejection area is in only one tail

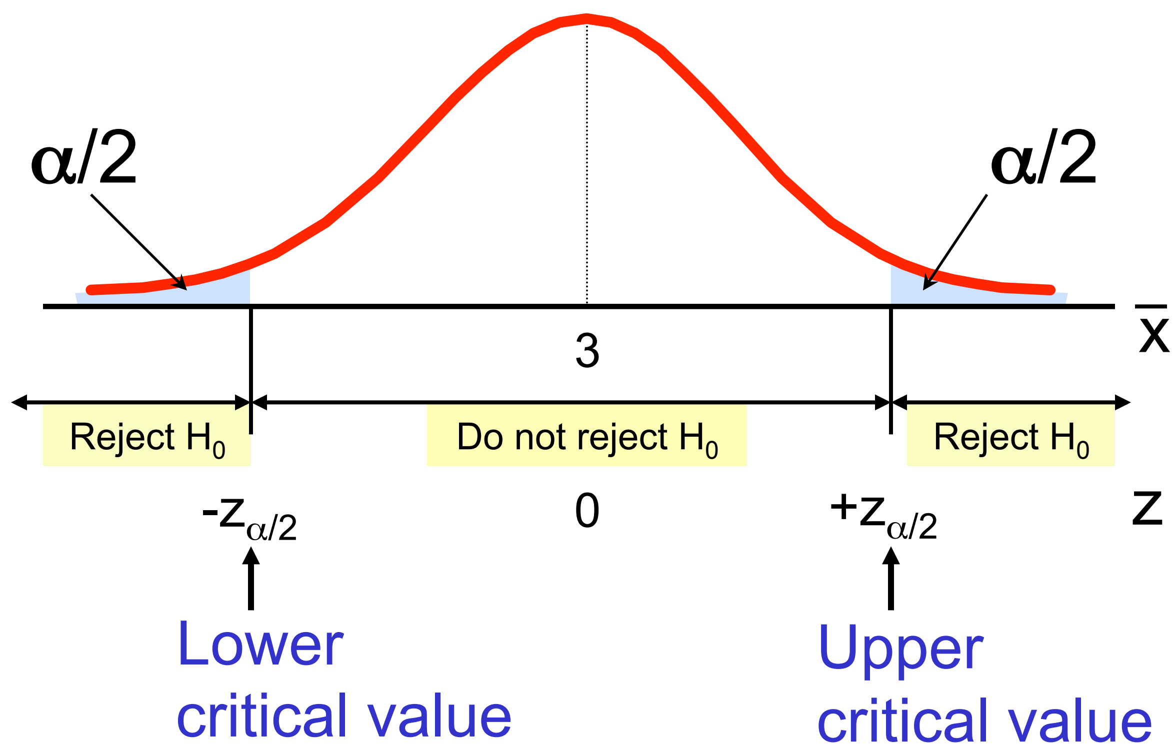


Two-Tail Tests

- In some settings, the alternative hypothesis does not specify a unique direction

H_0: $\mu = 3$
H_1: $\mu \neq 3$

- There are two critical values, defining the two regions of rejection



Hypothesis Testing Example

Test the claim that the true mean # of TV sets in US homes is not equal to 3. (Assume σ = 0.8)

- State the appropriate null and alternative hypotheses
  - H_0: μ = 3 , H_1: μ ≠ 3 (This is a two tailed test)
- Specify the desired level of significance
  - Suppose that α = .05 is chosen for this test
- Choose a sample size
  - Suppose a sample of size n = 100 is selected

Hypothesis Testing Example

(continued)

- Determine the appropriate technique
  - σ is known so this is a z test
- Set up the critical values
  - For α = .05 the critical z values are ±1.96
- Collect the data and compute the test statistic
  - Suppose the sample results are

    $n = 100$, $\bar{x} = 2.84$ ($\sigma = 0.8$ is assumed known)

So the test statistic is:

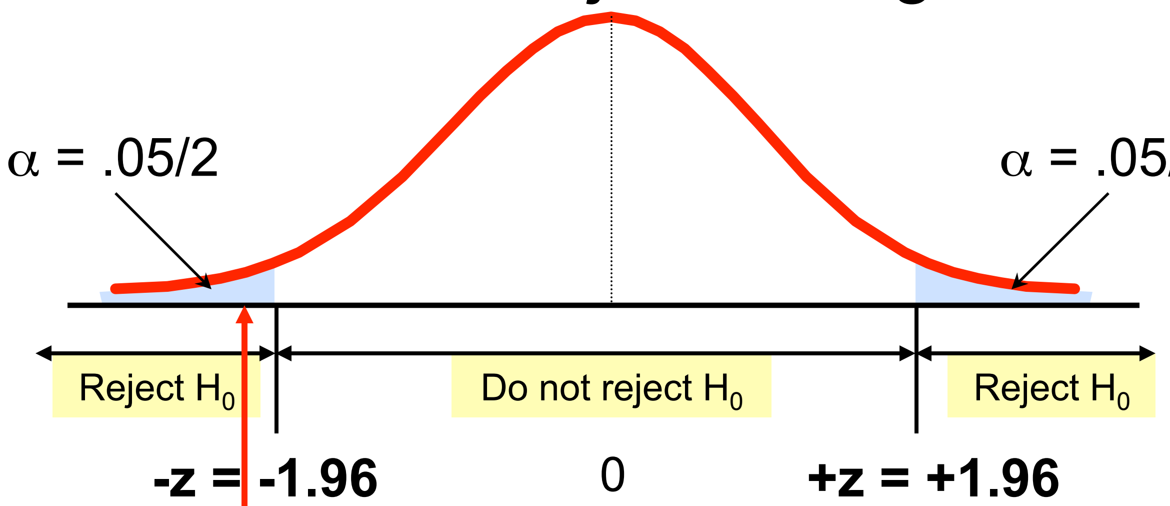
$$z = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{2.84 - 3}{\frac{0.8}{\sqrt{100}}} = \frac{-.16}{.08} = -2.0$$

Hypothesis Testing Example

(continued)

- Is the test statistic in the rejection region?

Reject H_0 if $z \le -1.96$ or $z \ge 1.96$; otherwise do not reject H_0

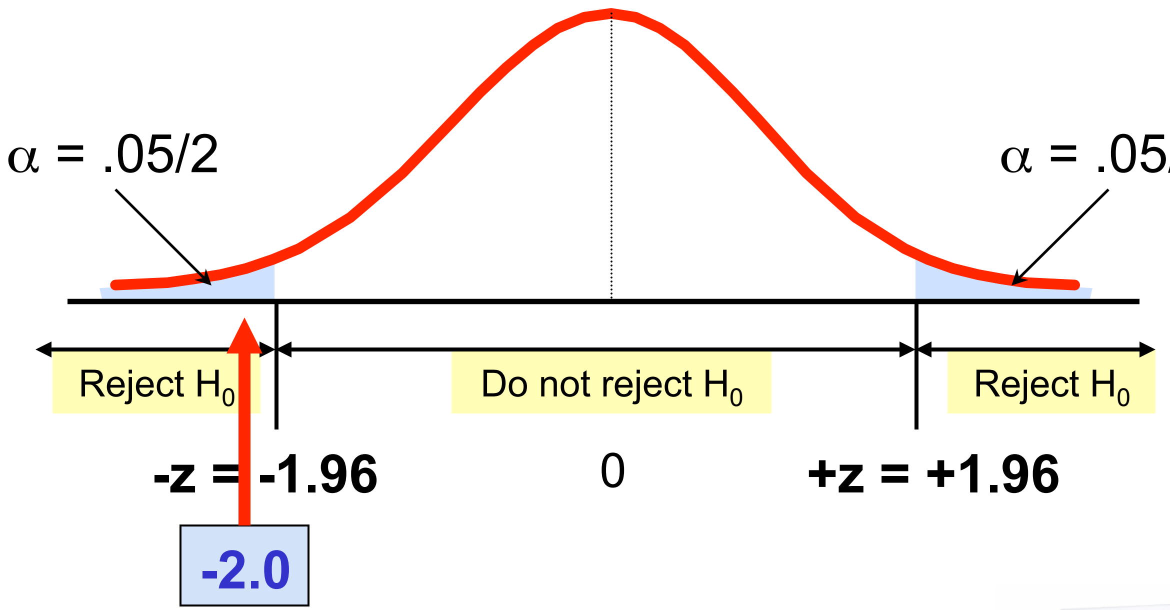


Here, $z = -2.0 < -1.96$, so the test statistic is in the rejection region

Hypothesis Testing Example

(continued)

- Reach a decision and interpret the result



Since $z = -2.0 < -1.96$, we reject the null hypothesis and conclude that there is sufficient evidence that the mean number of TVs in US homes is not equal to 3

Example: p-Value

- Example: How likely is it to see a sample mean of 2.84 (or something further from the mean, in either direction) if the true mean is $\mu = 3.0$?

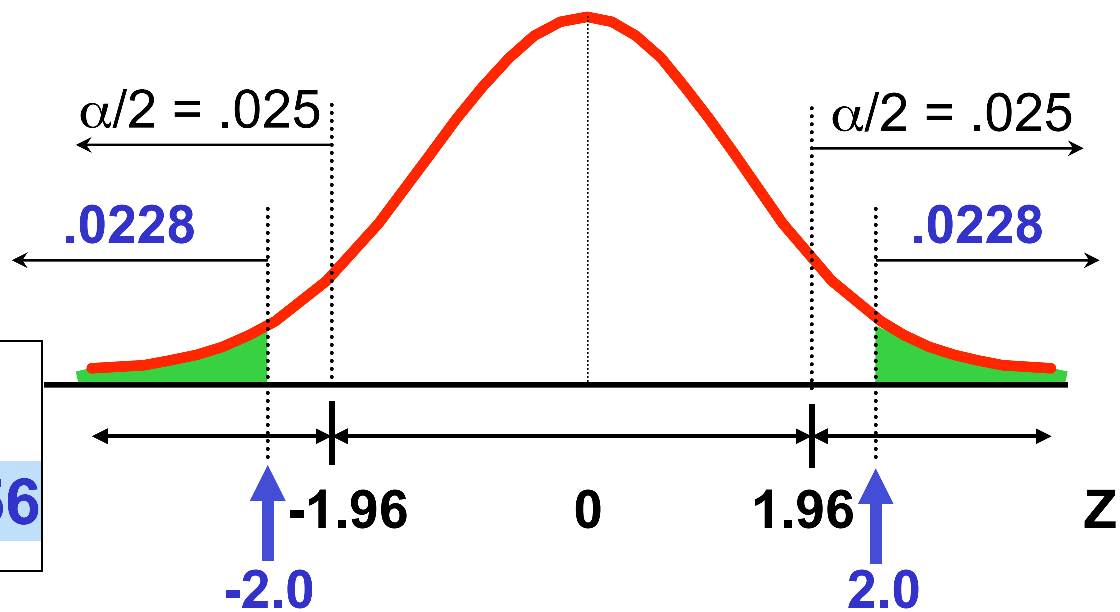
$\bar{x} = 2.84$ is translated to a z score of z = -2.0

$P(z < -2.0) = .0228$

$P(z > 2.0) = .0228$

p-value

= .0228 + .0228 = .0456



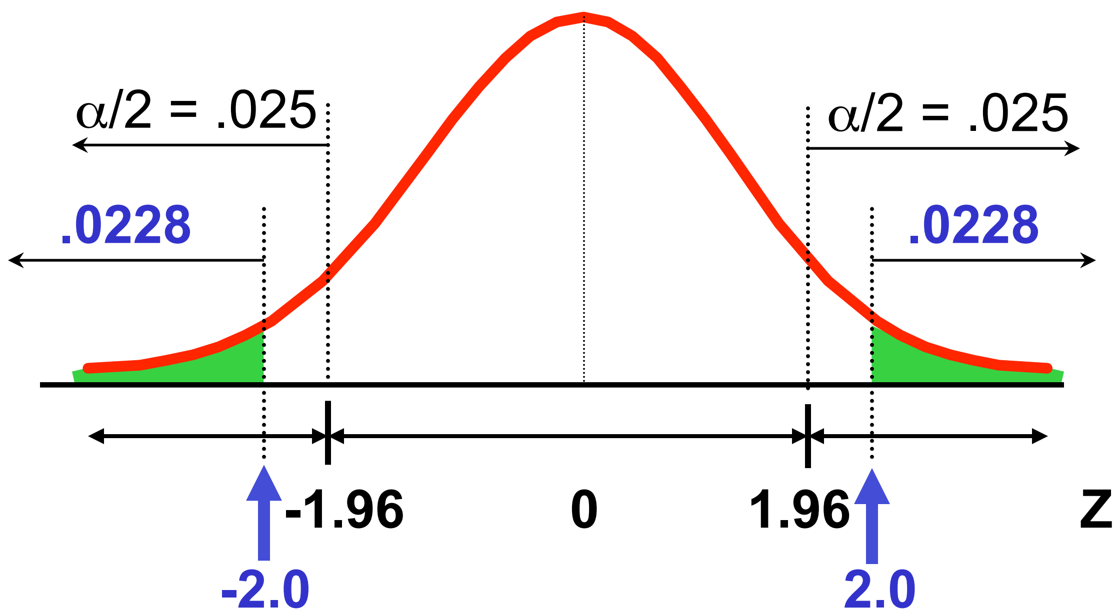
Example: p-Value

(continued)

- Compare the p-value with α
  - If p-value $\leq \alpha$, reject H_0
  - If p-value $> \alpha$, do not reject H_0

Here: p-value = .0456
α = .05

Since .0456 < .05, we reject the null hypothesis



t Test of Hypothesis for the Mean (σ Unknown)

9.3

- Convert sample result ($\bar{x}$) to a t test statistic

Hypothesis Tests for μ

- σ Known
- σ Unknown

Consider the test

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu > \mu_0$$

(Assume the population is normal)

The decision rule is:

$$\text{Reject } H_0 \text{ if } t = \frac{\bar{x} - \mu_0}{s / \sqrt{n}} \geq t_{n-1,\,\alpha}$$

t Test of Hypothesis for the Mean (σ Unknown)

(continued)

- For a two-tailed test:

Consider the test

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu \neq \mu_0$$

(Assume the population is normal, and the population variance is unknown)

The decision rule is:

$$\text{Reject } H_0 \text{ if } \boxed{t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \leq -t_{n-1\,,\,\alpha/2}} \text{ or if } \boxed{t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \leq t_{n-1\,,\,\alpha/2}}$$

Example: Two-Tail Test (σ Unknown)

The average cost of a hotel room in Chicago is said to be $168 per night. A random sample of 25 hotels resulted in $\bar{x}$ = $172.50 and s = $15.40. Test at the α = 0.05 level.

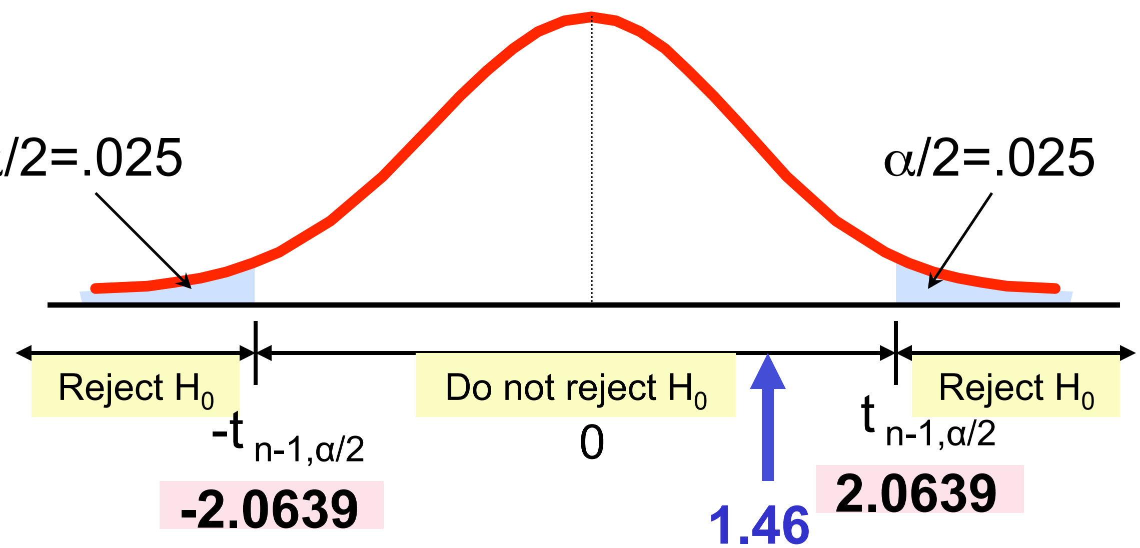
(Assume the population distribution is normal)

H_0: μ = 168

H_1: $\mu \neq$ 168

Example Solution: Two-Tail Test

H_0: μ = 168
H_1: μ ≠ 168



- α = 0.05
- n = 25
- σ is unknown, so use a **t statistic**
- Critical Value:

$t_{24\ ,\ .025} = \pm\ 2.0639$

$$t_{n-1} = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \frac{172.50 - 168}{\frac{15.40}{\sqrt{25}}} = 1.46$$

Do not reject H_0: not sufficient evidence that true mean cost is different than $168

9.4 Tests of the Population Proportion

- Involves categorical variables
- Two possible outcomes
  - "Success" (a certain characteristic is present)
  - "Failure" (the characteristic is not present)
- Fraction or proportion of the population in the "success" category is denoted by p
- Assume sample size is large

Proportions

(continued)

- Sample proportion in the success category is denoted by $\hat{p}$

  - $$\hat{p} = \frac{\text{number of successes in sample}}{\text{sample size}}$$

- When sample size is large, $\hat{p}$ can be approximated by a normal distribution with mean and standard deviation

  - $$\mu_{\hat{p}} = p \qquad \sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$$

Hypothesis Tests for Proportions

- The sampling distribution of $\hat{p}$ is approximately normal, so the test statistic is a z value:

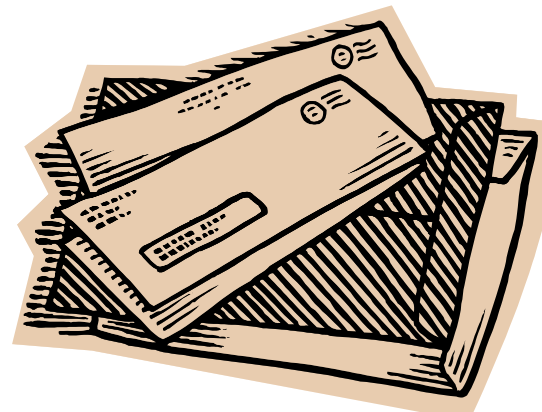
$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

Hypothesis Tests for p

- **Large sample**
- **Small sample**
  - Not discussed in this chapter

Example: Z Test for Proportion

A marketing company claims that it receives 8% responses from its mailing. To test this claim, a random sample of 500 were surveyed with 25 responses. Test at the α = .05 significance level.



Z Test for Proportion: Solution

H_0: p = .08
H_1: p ≠ .08

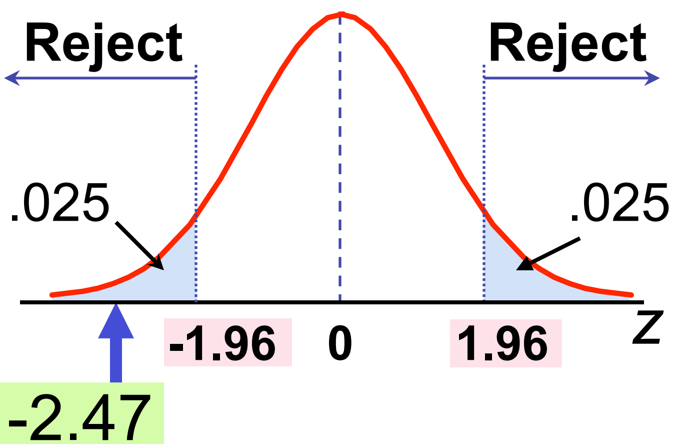
α = .05

n = 500, $\hat{p}$ = .05

Test Statistic:

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{.05 - .08}{\sqrt{\frac{.08(1-.08)}{500}}} = -2.47$$

Critical Values: ± 1.96



Decision:

Reject H_0 at α = .05

Conclusion:

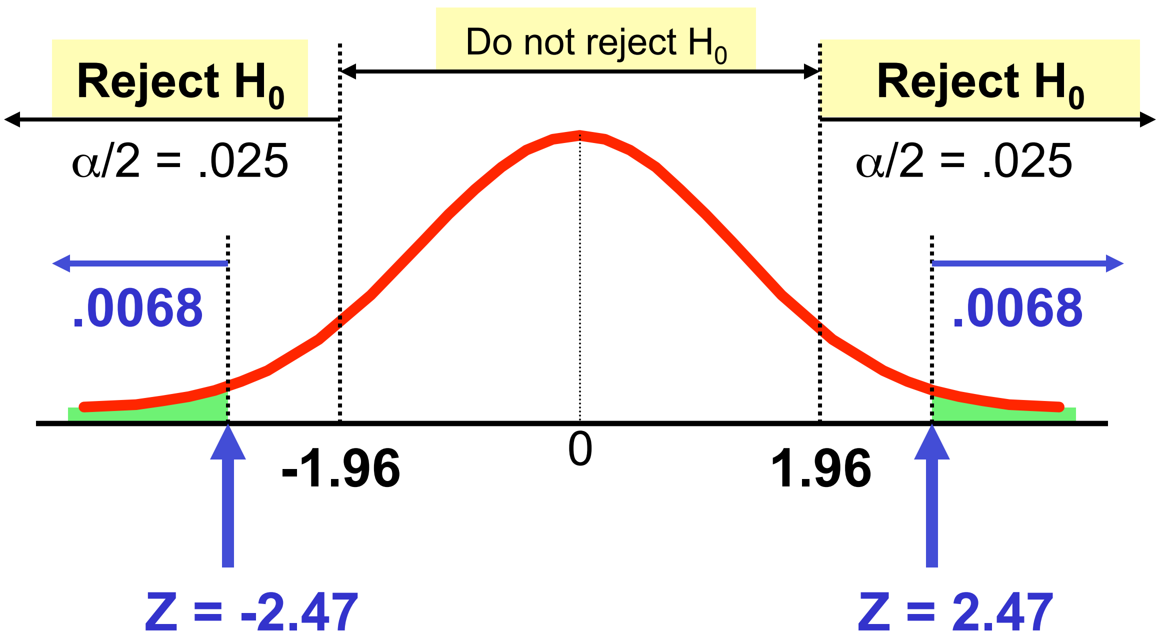
There is sufficient evidence to reject the company's claim of 8% response rate.

p-Value Solution

(continued)

Calculate the p-value and compare to α

(For a two sided test the p-value is always two sided)



p-value = .0136:

$$P(Z \leq -2.47) + P(Z \geq 2.47)$$

$$= 2(.0068) = 0.0136$$

Reject H_0 since p-value = .0136 < α = .05

Power of the Test

- Recall the possible hypothesis test outcomes:

Key:
Outcome
(Probability)

	Actual Situation	
Decision	H_0 True	H_0 False
Do Not Reject H_0	**No error** **(1 - α)**	**Type II Error** **(β)**
Reject H_0	**Type I Error** **(α)**	**No Error** **(1 - β)**

- β denotes the probability of Type II Error
- $1 – \beta$ is defined as the power of the test

Power = $1 – \beta$ = the probability that a false null hypothesis is rejected

Type II Error

Assume the population is normal and the population variance is known. Consider the test

$$H_0 : \mu \geq \mu_0 = 52$$
$$H_1 : \mu < \mu_0 = 52$$

The decision rule is:

$$\text{Reject } H_0 \text{ if } z = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \leq z_\alpha$$

or

$$\text{Reject } H_0 \text{ if } \bar{x} = \bar{x}_c \leq \mu_0 + z_\alpha \sigma / \sqrt{n}$$

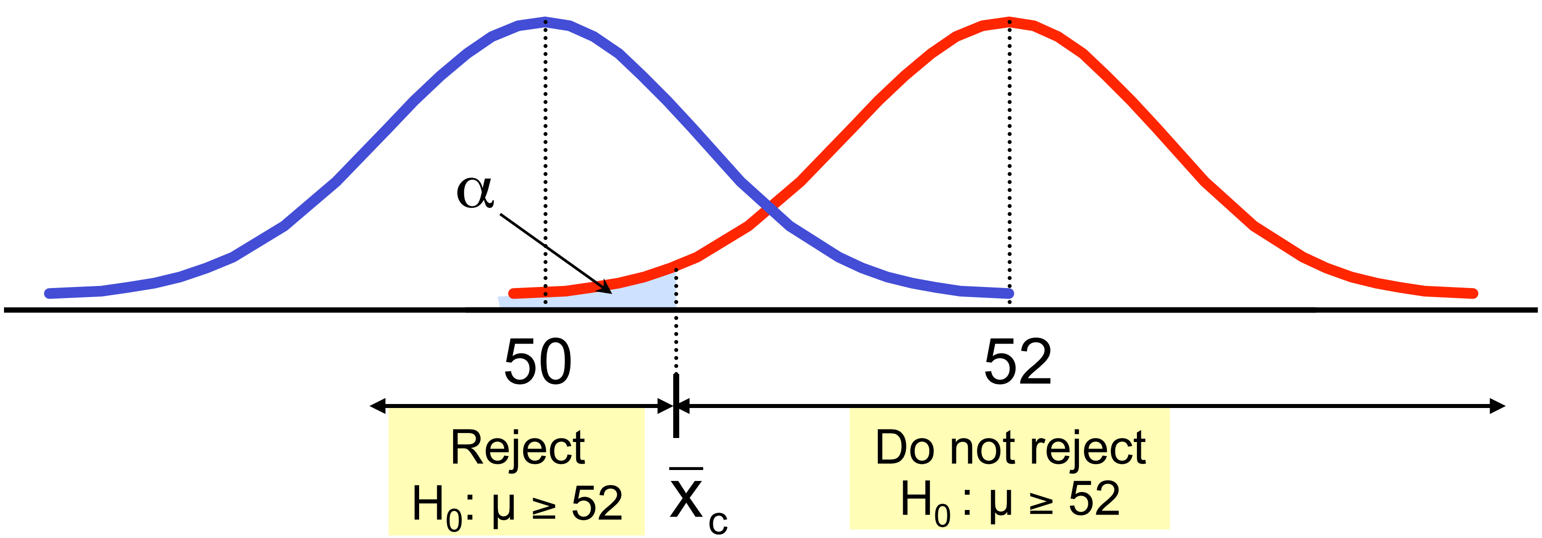
If the null hypothesis is false and the true mean is μ*, then the probability of type II error is

$$\beta = P(\bar{x} > \bar{x}_c \mid \mu = \mu^*) = P\left(z > \frac{\bar{x}_c - \mu^*}{\sigma / \sqrt{n}} \right)$$

Type II Error Example

- Type II error is the probability of failing to reject a false H_0

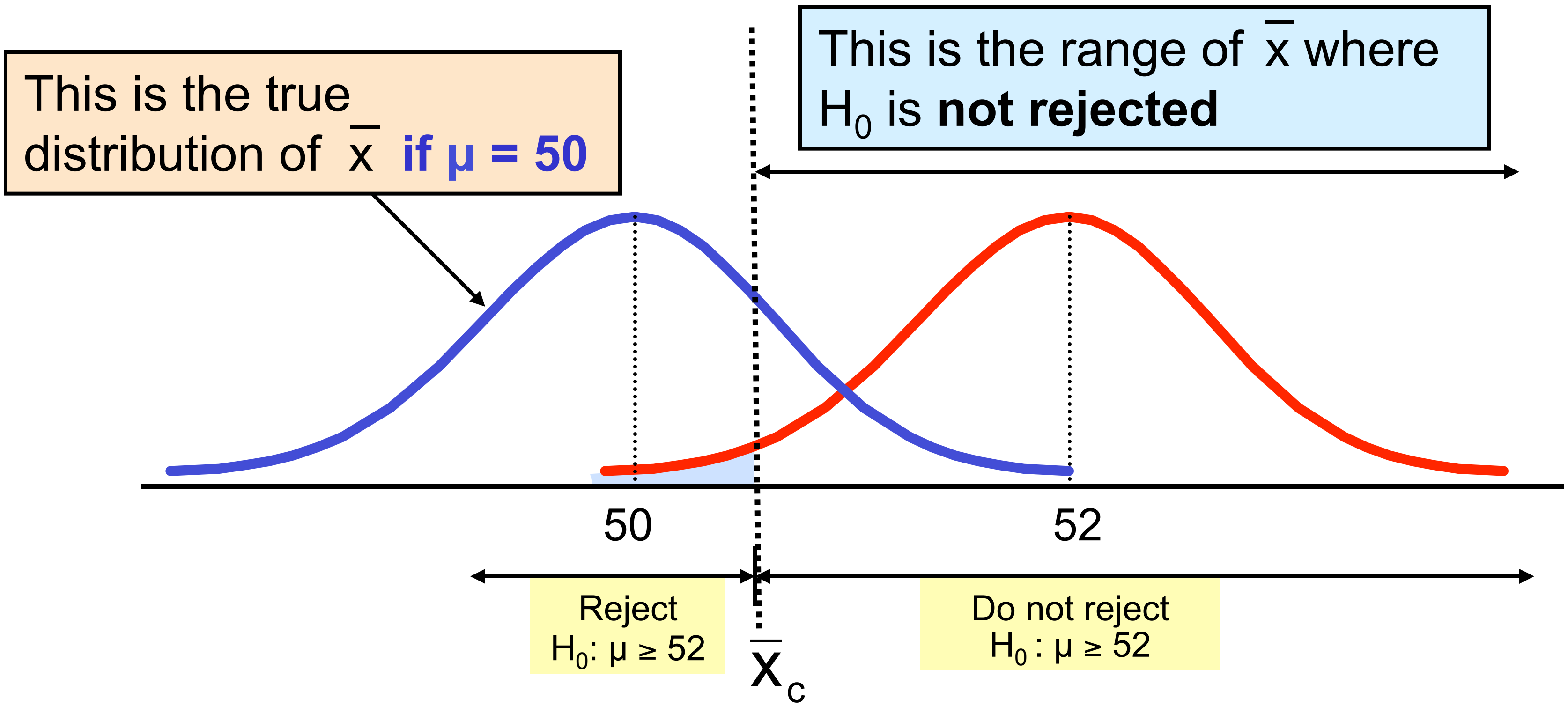
Suppose we fail to reject **H_0: $\mu \geq 52$**
when in fact the true mean is **$\mu^* = 50$**



Type II Error Example

(continued)

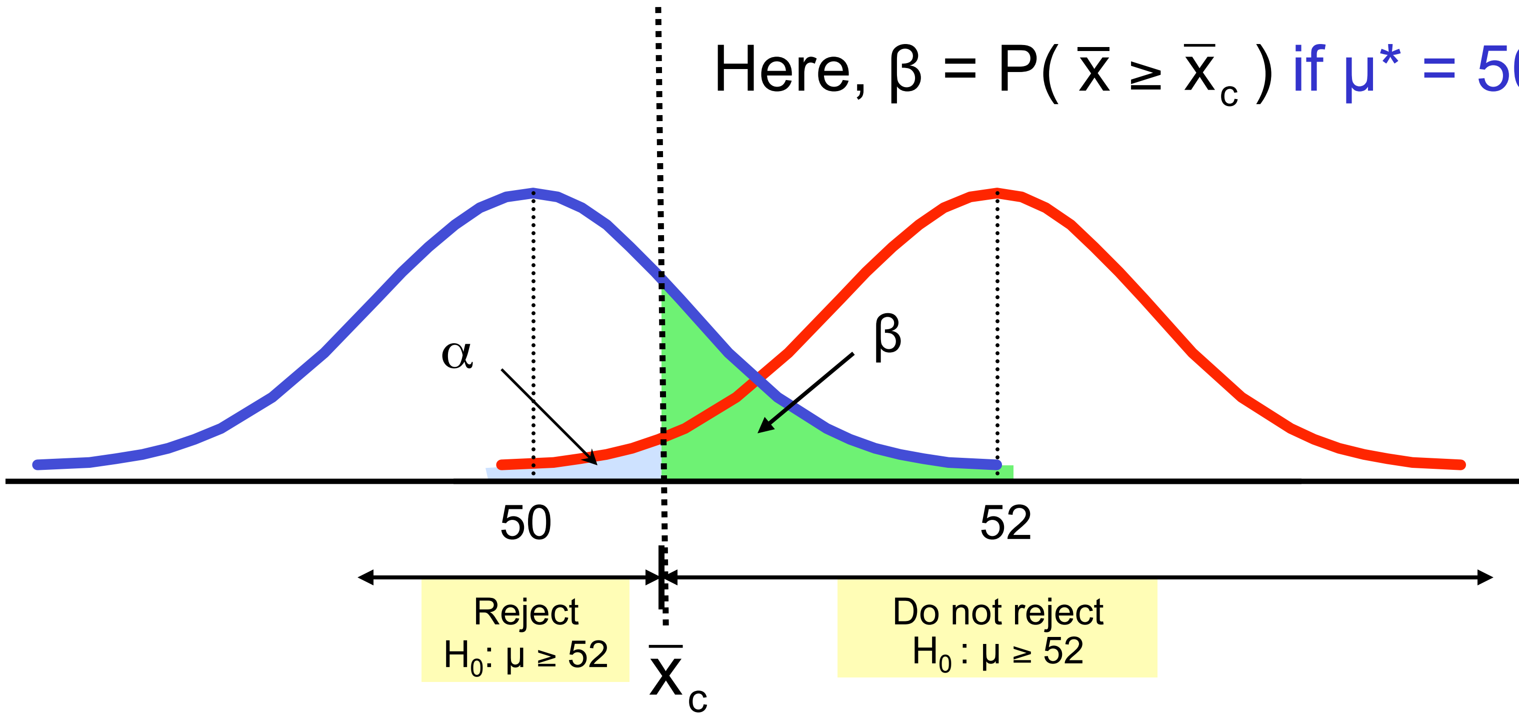
- Suppose we do not reject H_0: $\mu \geq 52$ when in fact the true mean is $\mu^* = 50$



Type II Error Example

(continued)

- Suppose we do not reject H_0: $\mu \geq 52$ when in fact the true mean is $\mu^* = 50$



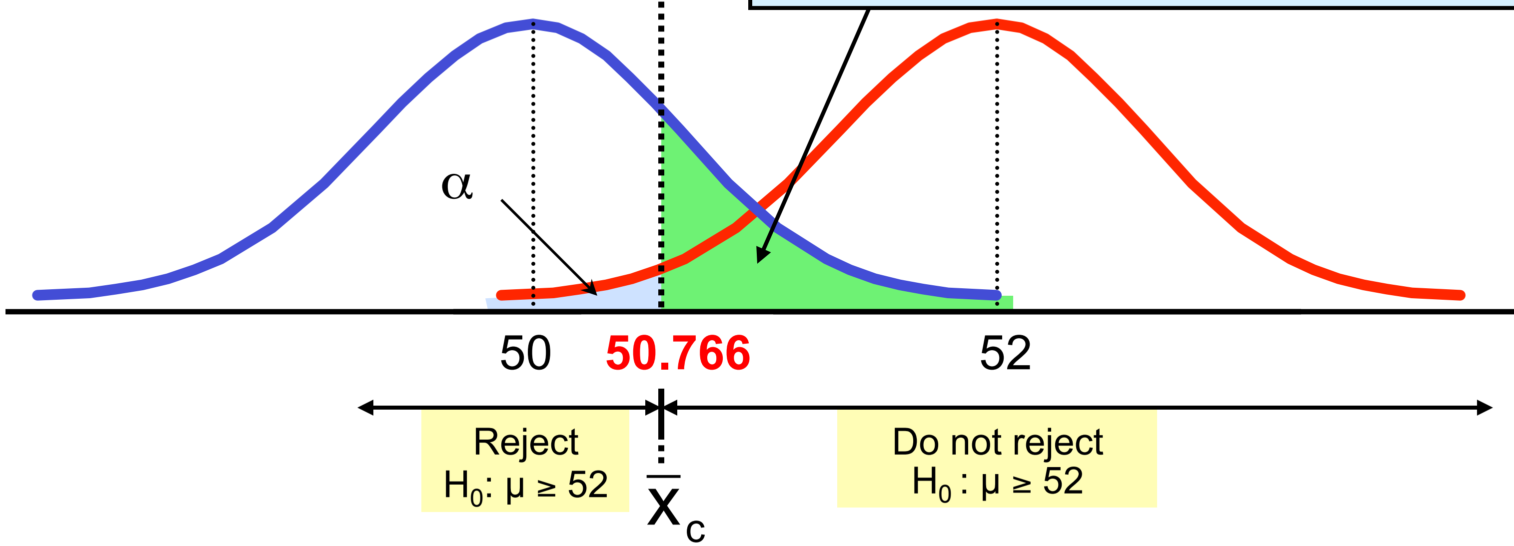
Calculating β

- Suppose n = 64 , σ = 6 , and α = .05

$$\bar{x}_c = \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} = 52 - 1.645 \frac{6}{\sqrt{64}} = \boxed{50.766}$$

(for $H_0 : \mu \geq 52$)

So $\beta = P(\bar{x} \geq 50.766)$ if $\mu^* = 50$

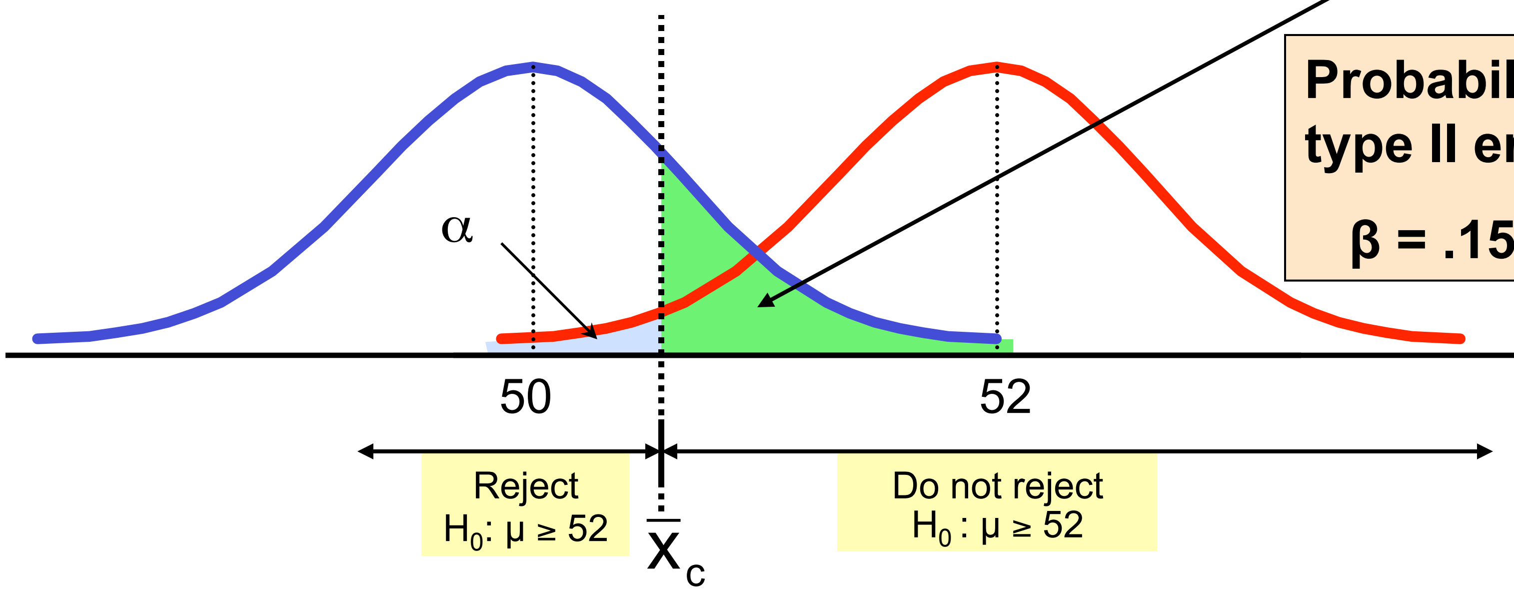


Calculating β

(continued)

- Suppose n = 64 , σ = 6 , and α = .05

$$P(\bar{x} \geq 50.766 \mid \mu^* = 50) = P\left(z \geq \frac{50.766 - 50}{6/\sqrt{64}}\right) = P(z \geq 1.02) = .5 - .3461 = \boxed{.1539}$$



Probability of type II error:

β = .1539

Power of the Test Example

If the true mean is $\mu^* = 50$,

- The probability of Type II Error = $\beta = 0.1539$
- The power of the test = $1 - \beta = 1 - 0.1539 = 0.8461$

Key:
Outcome
(Probability)

	Actual Situation	
Decision	H_0 True	H_0 False
Do Not Reject H_0	**No error** **$1 - \alpha = 0.95$**	**Type II Error** **$\beta = 0.1539$**
Reject H_0	**Type I Error** **$\alpha = 0.05$**	**No Error** **$1 - \beta = 0.8461$**

(The value of β and the power will be different for each μ^*)

Hypothesis Tests of one Population Variance

9.6

- Goal: Test hypotheses about the population variance, σ^2
  - If the population is normally distributed,

$$\boxed{\chi^2_{n-1} = \frac{(n-1)s^2}{\sigma^2}}$$

has a chi-square distribution with (n – 1) degrees of freedom

Hypothesis Tests of one Population Variance

(continued)

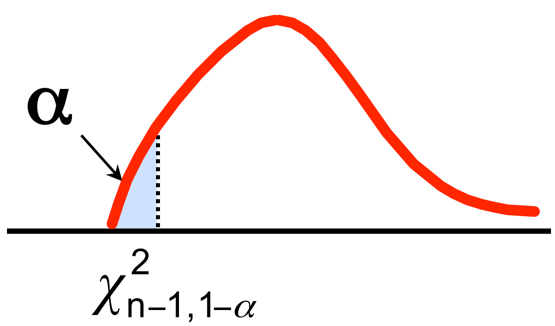
The test statistic for hypothesis tests about one population variance is

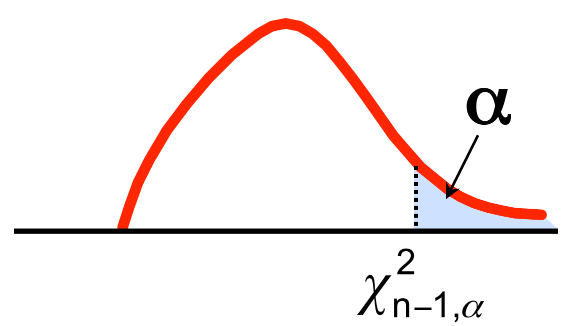
$$\chi^2_{n-1} = \frac{(n-1)s^2}{\sigma_0^2}$$

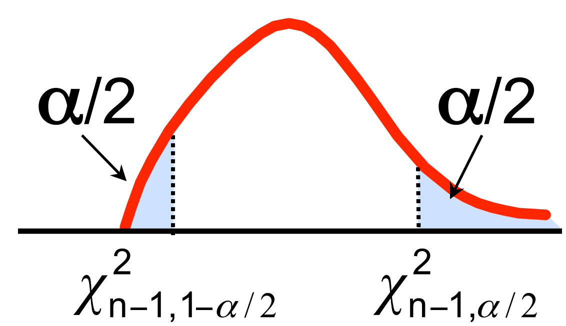
Decision Rules: Variance

Population variance

Lower-tail test:	Upper-tail test:	Two-tail test:
$H_0: \sigma^2 \geq \sigma_0^2$	$H_0: \sigma^2 \leq \sigma_0^2$	$H_0: \sigma^2 = \sigma_0^2$
$H_1: \sigma^2 < \sigma_0^2$	$H_1: \sigma^2 > \sigma_0^2$	$H_1: \sigma^2 \neq \sigma_0^2$







Reject H_0 if

$$\chi^2_{n-1} \leq \chi^2_{n-1,1-\alpha}$$

Reject H_0 if

$$\chi^2_{n-1} \geq \chi^2_{n-1,\alpha}$$

Reject H_0 if

$$\chi^2_{n-1} \geq \chi^2_{n-1,\alpha/2}$$

or

$$\chi^2_{n-1} \leq \chi^2_{n-1,1-\alpha/2}$$