Econ 325: Introduction to Empirical Economics

Lecture 3

Discrete Random Variables and Probability Distributions

4.1

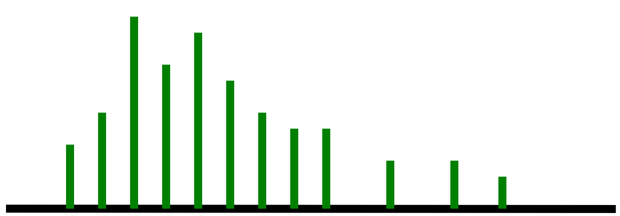
Introduction to Probability Distributions

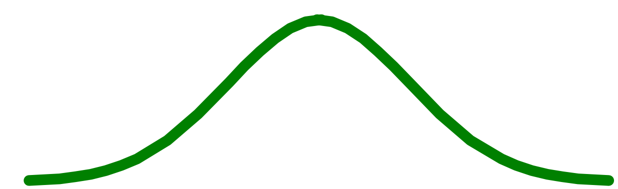
- **Random Variable**
  - Represents a possible numerical value from a random experiment

Random Variables

Ch. 4 **Discrete Random Variable**

Continuous Random Variable Ch. 5

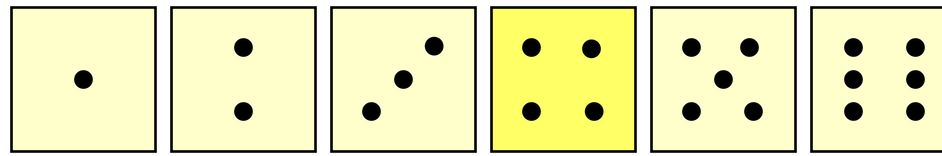




Discrete Random Variables

- Can only take on a countable number of values

Examples:



- **Roll a die twice**
  Let X be the number of times 4 comes up
  (then X could be 0, 1, or 2 times)

- **Toss a coin 3 times.**
  Let X be the number of heads
  (then X = 0, 1, 2, or 3)

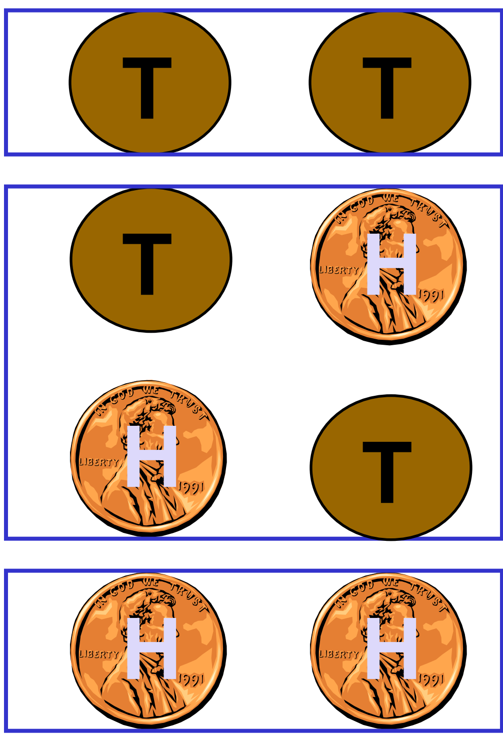
4.2

Discrete Probability Distribution

Experiment: Toss 2 Coins. Let X = # heads.

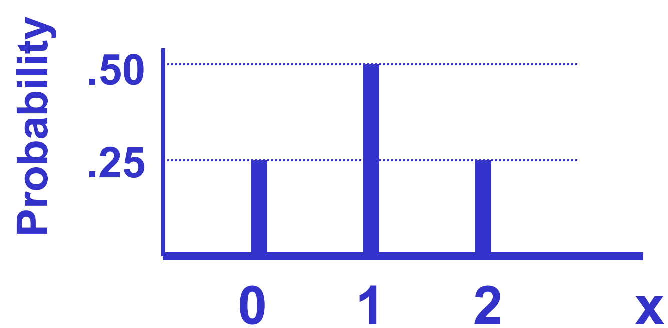
Show P(x) , i.e., P(X = x) , for all values of x:

4 possible outcomes



Probability Distribution

x Value	Probability
0	1/4 = .25
1	2/4 = .50
2	1/4 = .25



Random variable

- $S = \{TT, TH, HT, TH\}$
- Define a function $X(s)$ by

  $X(\{TT\})=0$, $X(\{TH\})=1$, $X(\{HT\})=1$, $X(\{HH\})=2$

- $P(X=0) = P(\{TT\}) = 1/4$
- $P(X=1) = P(\{TH,HT\}) = 1/2$

Definition: Random variable

- A random variable X is a function which maps the outcome of an experiment s to the real number x.

$$X\colon S \;\rightarrow \mathit{the\ space\ of\ X}$$

- The space of X is given by

$$S_X = \{x : X(s) = x, s \in S\}$$

Discrete Probability Distribution

- The space of X = {0,1,2}.
- Define a set A = {0,1} in the space of X. Then,

$$P(X \in A) = \sum_{x \in A} P(X = x) = P(X = 0) + P(X = 1)$$

- Notation: Uppercase ``X'' represents a random variable and lowercase ``x'' represents some constant (e.g., realized value).

Definition: Probability mass function

The probability mass function (pmf) of a discrete random variable X is a function that satisfies the following properties:

1). $\mathbf{0 \leq f}_X(\boldsymbol{x}) \leq \mathbf{1}$

2). $\sum_{\boldsymbol{x} \in S_X} \mathbf{f}_X(\boldsymbol{x}) = 1$

3). $\boldsymbol{P}(\boldsymbol{X} \in \boldsymbol{A}) = \sum_{\boldsymbol{x} \in \boldsymbol{A}} \mathbf{f}_X(\boldsymbol{x})$

Probability mass function

$$\boxed{f_X(x) = P(X = x)}$$

Cumulative Distribution Function

- The cumulative distribution function, denoted $F(x_0)$, is a function defined by the probability of X being less than or equal to x_0

$$F(x_0) = P(X \le x_0) = \sum_{x \le x_0} f_X(x)$$

Question

- Define X = # of heads when you toss 2 coins.
- What is the probability mass function and the cumulative distribution function of X?

Question

- Define X = a number you get from rolling a die.
- What is the probability mass function and the cumulative distribution function of X?

Expected Value

- Expected Value (or mean) of a discrete distribution

$$\mu = E(X) = \sum_{x} x f_X(x)$$

Example

- **What is the expected value when you roll a die once?**

- $f_X(i) = \mathrm{P}(X = i) = \frac{1}{6}$ **for** $i = 1, 2, \dots, 6$

- $E[X] = \sum_{i=1}^{6} i \times \left(\frac{1}{6}\right) = 3.5$

Clicker Question 3-1

Define X = # of heads when you toss 2 coins.

What is the expected value of X?

A). 0.5
B). 1
C). 1.5

Variance and Standard Deviation

- Variance of a discrete random variable X

$$\sigma^2 = E(X-\mu)^2 = \sum_x (x-\mu)^2 f_X(x)$$

- Standard Deviation of a discrete random variable X

$$\sigma = \sqrt{\sigma^2} = \sqrt{\sum_x (x-\mu)^2 f_X(x)}$$

Standard Deviation Example

- Example: Toss 2 coins, X = # heads, compute standard deviation (recall E(x) = 1)

$$\sigma = \sqrt{\sum_x (x-\mu)^2 f_X(x)}$$

$$\sigma = \sqrt{(0-1)^2(.25)+(1-1)^2(.50)+(2-1)^2(.25)} = \sqrt{.50} = .707$$

Possible number of heads = 0, 1, or 2

Clicker Question 3-2

- **Toss 1 coin. Let X = 1 if it is head and X=0 if it is tail. What is the variance of this random variable?**

A). 1

B). 0.5

C). 0.25

D). 0.1

Functions of Random Variables

- If P(x) is the probability function of a discrete random variable X , and g(X) is some function of X , then the expected value of function g is

$$\boxed{E[g(X)] = \sum_{x} g(x) f_X(x)}$$

Clicker Question 3-3

- Toss 1 coin. Let X = 1 if it is head and X=0 if it is tail. Consider a function g(X) such that g(1) = 100 and g(0) = 0. What is E[g(X)]?

A). 0

B). 100

C). 50

D). 10

Linear Functions of Random Variables

- Let a and b be any constants.
- a) $\boxed{E(a) = a \qquad \text{and} \qquad Var(a) = 0}$

  i.e., if a random variable always takes the value a, it will have mean a and variance 0

- b) $\boxed{E(bX) = bE(X) \qquad \text{and} \qquad Var(bX) = b^2 Var(X)}$

  i.e., the expected value of $b \cdot X$ is $b \cdot E(X)$

Linear Functions of Random Variables

(continued)

- Let random variable X have mean μ_x and variance σ^2_x
- Let a and b be any constants.
- Let Y = a + bX
- Then the mean and variance of Y are

$$\boxed{E(Y) = E(a + bX) = a + bE(X)}$$

$$\boxed{Var(Y) = Var(a + bX) = b^2 Var(X)}$$

- so that the standard deviation of Y is

$$\boxed{\sigma_Y = |b|\sigma_X}$$

Bernoulli Distribution

- Consider only two outcomes: “success” or “failure”
- Let p denote the probability of success
- Let $1 – p$ be the probability of failure
- Define random variable X:

  $X = 1$ if success, $X = 0$ if failure
- Then the Bernoulli probability function is

$$\boxed{P(X=0)=(1-p) \quad \text{and} \quad P(X=1)=p}$$

Possible Bernoulli Distribution Settings

- A survey responses of ``I will vote for the Liberal Party'' or ``I will vote for the Conservative Party''
- A manufacturing plant labels items as either defective or acceptable
- A marketing research firm receives survey responses of “yes I will buy” or “no I will not”

Bernoulli Distribution Mean and Variance

- The mean is $\boxed{\mu = p}$

$$\mu = E(X) = \sum_{x=0,1} xP(X = x) = (0)(1-p) + (1)p = p$$

- The variance is $\boxed{\sigma^2 = p(1 - p)}$

$$\sigma^2 = E[(X-\mu)^2] = \sum_{x=0,1} (x-\mu)^2 P(X = x)$$

$$= (0-p)^2(1-p) + (1-p)^2 p = p(1-p)$$

2019 Canadian federal election

Canada Poll Tracker: CBC News

338Canada

Nanos/CTV-G&M Polls, Sep 25

- Interview 1200 eligible voters by telephone from Sep 23-Sep 25.
- Out of 1200, 432 eligible voters say that they would vote for the Liberary Party.

Question

$X_i = 1$ if ``I will vote for the Liberal Party''

$p = P(X_i = 1) =$ the <u>population</u> fraction of voters who vote for the Liberal Party

Let X_1, X_2, and X_3 be survey responses from <u>randomly sampled</u> three individuals.

What is the probability mass function of

$Y = X_1 + X_2 + X_3$?

Question

Let X_i for $i = 1,2,\ldots,n$ are survey responses from randomly sampled n individuals with $P(X_i = 1) = p$.

What is the probability mass function of

$Y = \sum_{i=1}^{n} X_i$?

Binomial Distribution

Consider the **sum** of n independent Bernoulli random variables:

$$\boxed{Y = \sum_{i=1}^{n} X_i}$$

, where

$$\boxed{\mathrm{P}(\mathrm{X_i} = 0) = (1 - \mathrm{p}) \text{ and } \mathrm{P}(\mathrm{X_i} = 1) = \mathrm{p}}$$

P(Y=y) = probability of **y** successes in **n** trials,
with probability of success **p** on each trial

y = number of 'successes' in sample (y = 0, 1, 2, ..., n)

n = sample size (number of trials or observations)

p = probability of "success"

Probability mass function of Binomial distribution

$$P(Y=y) = \frac{n!}{y!(n-y)!} p^{y} (1-p)^{n-y}$$

P(y) = probability of **y** successes in **n** trials,
with probability of success **p** on each trial

y = number of 'successes' in sample,
(y = 0, 1, 2, ..., n)

n = sample size (number of trials
or observations)

p = probability of "success"

Clicker Question 3-4

Randomly sampled 3 individuals. What is the probability that 2 out of 3 person supports the Liberal party if p=0.4?

(A). $0.4 \times (1-0.4)^2$

(B). $(1-0.4)\times (0.4)^2$

(C). $3\times 0.4 \times (1-0.4)^2$

(D). $3\times(1-0.4)\times (0.4)^2$

Binomial Distribution Mean and Variance

- Mean

$$\mu = E(Y) = E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i) = np$$

- Variance and Standard Deviation

$$\sigma^2 = np(1-p)$$

$$\sigma = \sqrt{np(1-p)}$$

Average of n independent Bernoulli random variable

Consider the sample average of n independent Bernoulli random variable:

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$, with $$\mathrm{P(X_i = 1) = p \ \ and \ \ P(X_i = 0) = (1 - p)}$$

Then, $\bar{X}$ is related to Binomial random variable $Y = \sum_{i=1}^{n} X_i$ as

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{n} Y$$

Clicker Question 3-5

Consider the sample average of n independent Bernoulli random variable:

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$, with $$P(X_i = 1) = p \text{ and } P(X_i = 0) = (1 - p)$$

What is $E(\bar{X})$?

A). p
B). 1-p
C). np

Nanos/CTV-G&M Polls, Sep 25

- Interview 1200 eligible voters by telephone from Sep 23-Sep 25.
- 36 percent of eligible voters say that they would vote for the Liberary Party.
- $\bar{X} = 0.36$

Poisson Distribution Function

- The **Poisson probability distribution** gives the probability of a number of events occurring in a **fixed interval** of time or space.
- Examples:
  - The number of telephone calls to 911 in a large city from 1am to 5am.
  - The number of delivery trucks to arrive at a central warehouse in an hour.
  - The number of customers to arrive at a checkout aisle in your local grocery store from 2pm to 3pm.

Poisson Distribution Function

- Assume an interval is divided into a very large number of ``very short'' subintervals with equal length h.

1. The number of occurrences in subintervals are independent.
2. The probability of exactly one occurrence in a subinterval of length h is approximately λh.
3. The probability of two or more occurrences approaches zero as the length h approaches zero.

Poisson Distribution Function

- The expected number of occurrences per time/space unit is the parameter λ (lambda).

$$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

where:

P(x) = the probability of x occurrences over one unit of time or space

λ = the expected number of occurrences per time/space unit, $\lambda > 0$

e = base of the natural logarithm system (2.71828...)

Poisson Distribution Characteristics

Mean and variance of the Poisson distribution

- Mean

$$\mu_x = E[X] = \lambda$$

- Variance and Standard Deviation

$$\sigma_x^2 = E\left[\left(X - \mu_x\right)^2\right] = \lambda$$

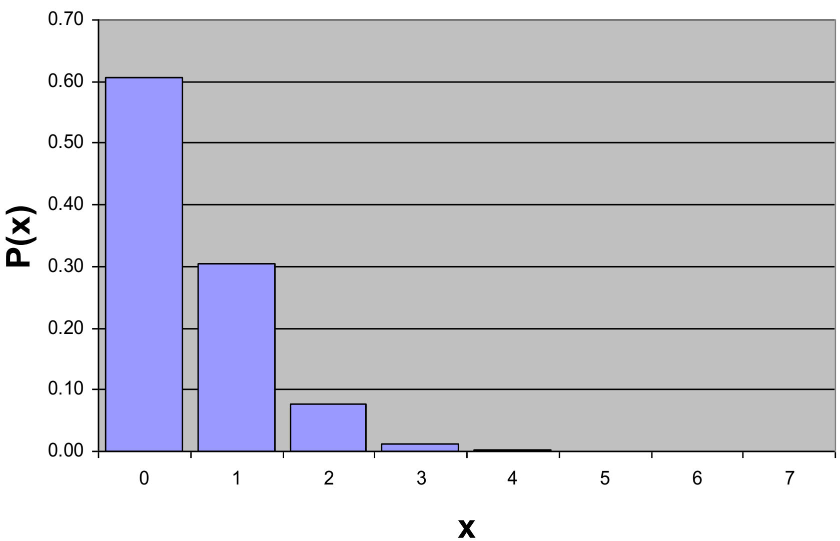
$$\sigma = \sqrt{\lambda}$$

where λ = expected number of occurrences per time/space unit

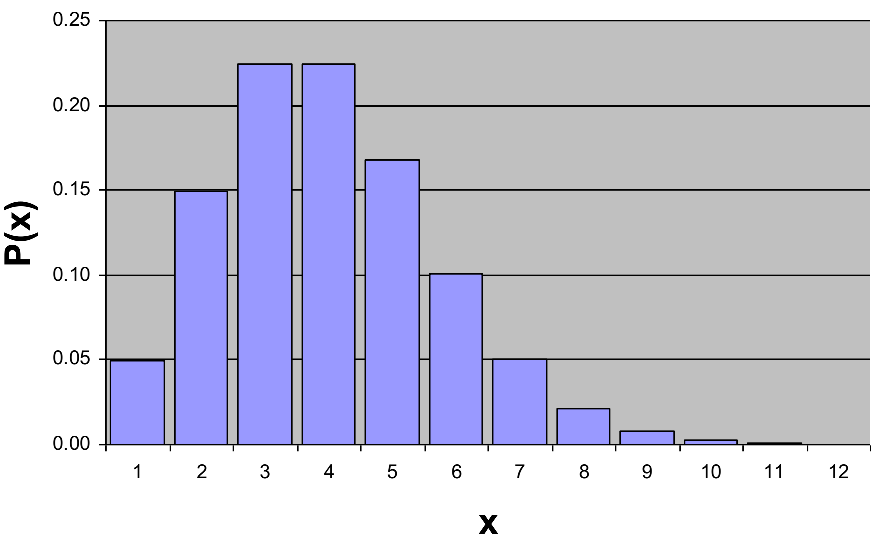
Poisson Distribution Shape

- The shape of the Poisson Distribution depends on the parameter λ :

λ = 0.50



λ = 3.00



Example

You are the CEO of a grocery store. Customers arrive at checkout counters at an average rate of 1 customer every 2 minutes. Assume that these arrivals are independent over time.

What is the probability that more than two customers arrive within one minute?

In this case, the expected number of customers per minute is $\lambda = 1/2 = 0.5$

Using Poisson Tables

X	λ								
	0.10	0.20	0.30	0.40	**0.50**	0.60	0.70	0.80	0.90
0	0.9048	0.8187	0.7408	0.6703	0.6065	0.5488	0.4966	0.4493	0.4066
1	0.0905	0.1637	0.2222	0.2681	0.3033	0.3293	0.3476	0.3595	0.3659
2	0.0045	0.0164	0.0333	0.0536	**0.0758**	0.0988	0.1217	0.1438	0.1647
3	0.0002	0.0011	0.0033	0.0072	0.0126	0.0198	0.0284	0.0383	0.0494
4	0.0000	0.0001	0.0003	0.0007	0.0016	0.0030	0.0050	0.0077	0.0111
5	0.0000	0.0000	0.0000	0.0001	0.0002	0.0004	0.0007	0.0012	0.0020
6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	0.0003
7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Example: Find P(X = 2) if $\lambda = .50$

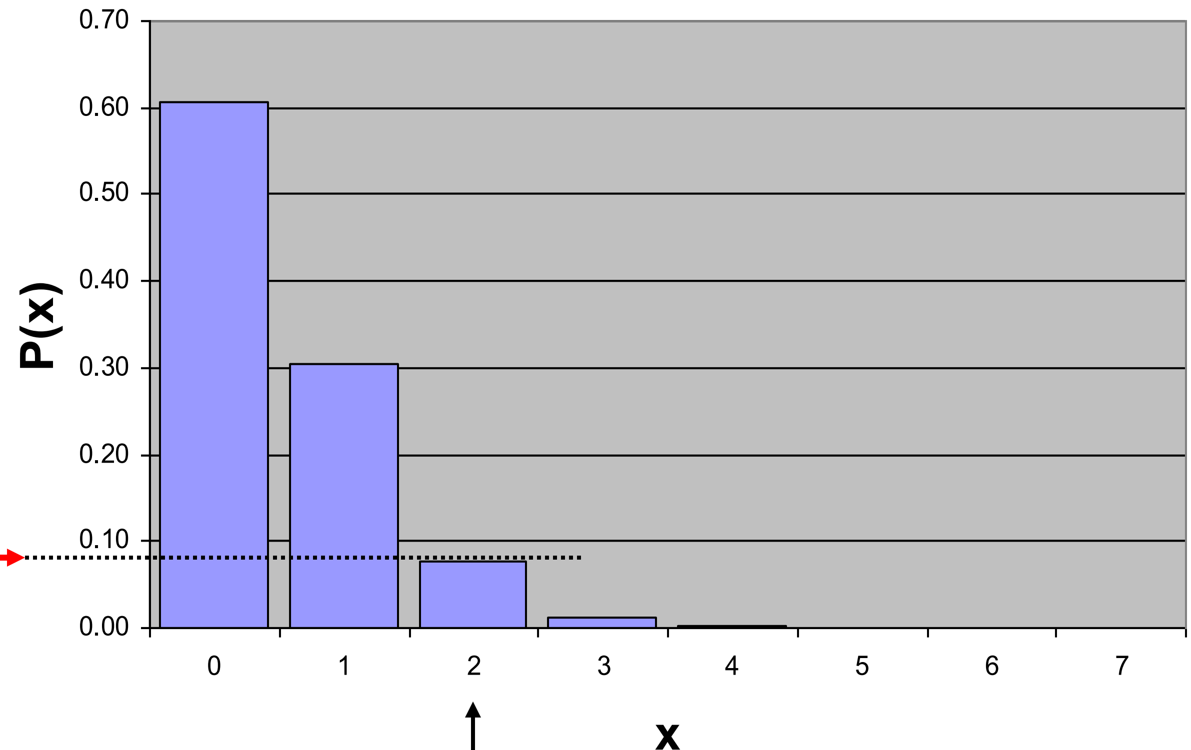
$$P(X=2) = \frac{e^{-\lambda}\lambda^X}{X!} = \frac{e^{-0.50}(0.50)^2}{2!} = .0758$$

Graph of Poisson Probabilities

Graphically:

λ = .50

X	**λ = 0.50**
0	0.6065
1	0.3033
2	0.0758
3	0.0126
4	0.0016
5	0.0002
6	0.0000
7	0.0000



P(X = 2) = .0758

Clicker Question 3-6

Customers independently arrive at counters at an average rate of 1 customer every 2 minutes.

What is the probability that more than two customers arrive within one minute?

A). 0.0758

B). 0.0886

C). 0.6065

Poisson and Binomial Distribution

- Divide one unit of time into n subintervals, each of which has length of $h = 1/n$.
- For sufficiently large n, the probability of one occurrence is given by $\lambda h = \lambda / n \Rightarrow$ a sequence of n Bernoulli trials.
- The number of occurrences within one unit of time is approximate by the sum of n Bernoulli trials, i.e., Binomial distribution:

$$P(X = x) \approx \frac{n!}{x!(n-x)!}\left(\frac{\lambda}{n}\right)^{x}\left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$\to \frac{\lambda e^{-\lambda}}{x!} \quad \text{as } n \to \infty$$

Poisson and Binomial Distribution

- Set $p = \lambda/n \Rightarrow \lambda = np$.
- Then, we may approximate the binomial distribution by the Poisson distribution:

$$P(X = x) = \frac{n!}{x!(n-x)!} p^x (1 - p)^{n-x}$$

$$\approx \frac{np e^{-np}}{x!} \quad \text{if n is large}$$

4.7 Joint probability mass functions

- A joint probability mass function is used to express the probability that X takes the specific value x and simultaneously Y takes the value y, as a function of x and y

$$f(x, y) = P(X = x, Y = y)$$

- The marginal probabilities are

$$f_X(x) = \sum_y f(x, y) \qquad f_Y(y) = \sum_x f(x, y)$$

Stochastic Independence

- The jointly distributed random variables X and Y are said to be independent if and only if

$$\boxed{f(x, y) = f_X(x) f_Y(y)}$$

for all possible pairs of values x and y

- A set of k random variables are independent if and only if

$$\boxed{f(x_1, x_2, \cdots, x_k) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_k}(x_k)}$$

Clicker Question 3-7

- Is X and Y stochastically independent?

	Y=30	Y=60	Y=100	Marginal Dist. of X
X=0	0.24	0.12	0.04	0.40
X=1	0.12	0.36	0.12	0.60
Marginal Dist. of Y	0.36	0.48	0.16	1.00

A). X and Y are independent

B). X and Y are not independent

Conditional probability mass functions

- The conditional probability mass function of the random variable Y is define by

$$f_{Y|X}(y \mid x) = \frac{f(x, y)}{f_X(x)}$$

- Similarly, the conditional probability mass function of X given Y = y is:

$$f_{X|Y}(x \mid y) = \frac{f(x, y)}{f_Y(y)}$$

Question

- What is the conditional probability mass function of Y given X=1?

	Y=30	Y=60	Y=100	Marginal Dist. of X
X=0	0.24	0.12	0.04	0.40
X=1	0.12	0.36	0.12	0.60
Marginal Dist. of Y	0.36	0.48	0.16	1.00

Conditional Mean and Variance

- The conditional mean is

$$\mu_{Y|X=x} = E_{Y|X}[Y \mid X = x] = \sum_{y} y\, f(y \mid x)$$

- E[Y|X=x] is a function of x and, therefore, is also called as ``conditional expectation function (CEF)''

- The conditional variance is

$$\sigma^2_{Y|X=x} = E_{Y|X}[(Y - \mu_{Y|X=x})^2 \mid X = x] = \sum_{y} (y - \mu_{Y|X=x})^2 f(y \mid x)$$

Clicker Question 3-8

- What is E[X|Y=30]?

	Y=30	Y=60	Y=100	Marginal Prob of X
X=0	0.24	0.12	0.04	0.40
X=1	0.12	0.36	0.12	0.60
Marginal Prob of Y	0.36	0.48	0.16	1.00

A). 1/2 B). 1/3 C). 2/3 D).1/4

Clicker Question 3-9

- What is Var[X|Y=30]?

	Y=30	Y=60	Y=100	Marginal Prob of X
X=0	0.24	0.12	0.04	0.40
X=1	0.12	0.36	0.12	0.60
Marginal Prob of Y	0.36	0.48	0.16	1.00

A). 1/9 B). 1/3 C). 2/9 D). 2/27

$E_{Y|X}[Y|X]$ as a random variable

- Viewing X as a random variable, $E_{Y|X}[Y|X]$ is a random variable because the value of $E_{Y|X}[Y|X]$ depends on a realization of X.

- The **Law of Iterated Expectations**:

$$E_X[E_{Y|X}[Y|X]] = E_Y[Y]$$

Covariance

- Let X and Y be discrete random variables with means μ_X and μ_Y
- The expected value of $(X - \mu_X)(Y - \mu_Y)$ is called the covariance between X and Y
- For discrete random variables

$$\boxed{\mathrm{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = \sum_x \sum_y (x-\mu_X)(y-\mu_Y)f(x,y)}$$

Correlation

- The correlation between X and Y is:

$$\boxed{\rho = \text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}}$$

- $\rho = 0$: no linear relationship between X and Y
- $\rho > 0$: positive linear relationship between X and Y
  - when X is high (low) then Y is likely to be high (low)
  - $\rho = +1$: perfect positive linear dependency
- $\rho < 0$: negative linear relationship between X and Y
  - when X is high (low) then Y is likely to be low (high)
  - $\rho = -1$: perfect negative linear dependency

Uncorrelatedness, Mean Independence, Stochastic Independence

- X and Y are said to be **uncorrelated** when Cov(X,Y)=0 or $\rho = 0$.

- X is said to be **mean independent** of Y when $E_{X|Y}[X|Y] = E_X[X]$.

- X and Y are said to be **stochastically independent** when $f(x, y) = f_X(x) f_Y(y)$.

Uncorrelatedness, Mean Independence, Stochastic Independence

Stochastic Independence

$$f(x,y) = f_X(x) f_Y(y)$$

$$\Downarrow$$

Mean Independence

$E_{X|Y}[X|Y] = E_X[X]$ or $E_{Y|X}[Y|X] = E_Y[Y]$

$$\Downarrow$$

Uncorrelatedness

Cov(X,Y)=0

Clicker Question 3-10

- Suppose X and Y are stochastically independent. Then,

A). the conditional mean of X given Y=y is the same as the unconditional mean of X.

B). the conditional mean of X given Y=y may not be the same as the unconditional mean of X.

$Var(aX + bY)$

- For any constant a and b and any two random variables X and Y,

$$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y)$$
$$+2ab\ Cov(X, Y)$$

Clicker Question 3-10

- Which of the following is true.

A). $Var(aX + bY) = a^2Var(X) + b^2Var(Y)$
$+2ab\ Corr(X,Y)$

B). $Var(aX + bY) = a^2Var(X) + b^2Var(Y)$
$+2ab\sigma_X\sigma_Y Corr(X,Y)$

C). $Var(aX + bY) = a^2Var(X) + b^2Var(Y)$
$+2abCorr(X,Y)/\sigma_X\sigma_Y$

Portfolio Analysis

- Let random variable X be the share price for stock A
- Let random variable Y be the share price for stock B
- The market value, W, for the portfolio is given by the linear function

$$\boxed{W = aX + bY}$$

- ``a'' and ``b'' are the numbers of shares of stock A and B, respectively.
- The return from holding the portfolio W:

$$\boxed{\Delta W = a\Delta X + b\Delta Y}$$

Portfolio Analysis

(continued)

- The mean value for ΔW is

$$E[\Delta W] = E[a\Delta X + b\Delta Y]$$
$$= aE[\Delta X] + bE[\Delta Y]$$

- The variance for ΔW is

$$\sigma^2_{\Delta W} = a^2\sigma^2_{\Delta X} + b^2\sigma^2_{\Delta Y} + 2ab\mathrm{Cov}(\Delta X, \Delta Y)$$

or using the correlation formula

$$\sigma^2_{\Delta W} = a^2\sigma^2_{\Delta X} + b^2\sigma^2_{\Delta Y} + 2ab\mathrm{Corr}(\Delta X, \Delta Y)\sigma_{\Delta X}\sigma_{\Delta Y}$$

Asset Class Correlation Matrix

Correlation Matrix: Daily % Change Correlation Over Last Ten Years

Ticker	S&P 500	C. Disc.	C. Stap	Energy	Finan.	H Care	Indust.	Mater.	Tech	Telcom	Utilit.	Oil	Gold	Dollar	L Bnd
S&P 500	1.00	0.94	0.85	0.81	0.88	0.86	0.94	0.89	0.90	0.78	0.76	0.24	-0.01	-0.17	-0.37
Cons. Disc.	0.94	1.00	0.81	0.68	0.82	0.78	0.90	0.82	0.85	0.72	0.67	0.16	-0.07	-0.12	-0.34
Cons. Stap.	0.85	0.81	1.00	0.65	0.67	0.81	0.78	0.71	0.70	0.69	0.71	0.12	-0.06	-0.10	-0.28
Energy	0.81	0.68	0.65	1.00	0.62	0.66	0.73	0.81	0.66	0.59	0.70	0.48	0.16	-0.28	-0.30
Financials	0.88	0.82	0.67	0.62	1.00	0.68	0.82	0.73	0.72	0.64	0.57	0.16	-0.06	-0.13	-0.31
H Care	0.86	0.78	0.81	0.66	0.68	1.00	0.78	0.71	0.73	0.68	0.69	0.14	-0.04	-0.12	-0.29
Industrials	0.94	0.90	0.78	0.73	0.82	0.78	1.00	0.87	0.84	0.70	0.68	0.21	-0.01	-0.18	-0.37
Materials	0.89	0.82	0.71	0.81	0.73	0.71	0.87	1.00	0.79	0.65	0.67	0.29	0.15	-0.27	-0.35
Technology	0.90	0.85	0.70	0.66	0.72	0.73	0.84	0.79	1.00	0.71	0.63	0.16	-0.04	-0.10	-0.35
Telecom	0.78	0.72	0.69	0.59	0.64	0.68	0.70	0.65	0.71	1.00	0.63	0.12	-0.05	-0.10	-0.25
Utilities	0.76	0.67	0.71	0.70	0.57	0.69	0.68	0.67	0.63	0.63	1.00	0.19	0.02	-0.15	-0.22
Oil	0.24	0.16	0.12	0.48	0.16	0.14	0.21	0.29	0.16	0.12	0.19	1.00	0.29	-0.30	-0.22
Gold	-0.01	-0.07	-0.06	0.16	-0.06	-0.04	-0.01	0.15	-0.04	-0.05	0.02	0.29	1.00	-0.43	0.07
Dollar	-0.17	-0.12	-0.10	-0.28	-0.13	-0.12	-0.18	-0.27	-0.10	-0.10	-0.15	-0.30	-0.43	1.00	-0.05
Long Bond	-0.37	-0.34	-0.28	-0.30	-0.31	-0.29	-0.37	-0.35	-0.35	-0.25	-0.22	-0.22	0.07	-0.05	1.00

Example: Investment Returns

Return per $100 for two types of investments

		Investment	
P(ΔX,ΔY)	**Economic condition**	**Bond Fund X**	**Aggressive Fund Y**
0.2	Recession	+ $ 7	- $20
0.5	Stable Economy	+ 4	+ 6
0.3	Expanding Economy	+ 2	+ 35

$$E(\Delta X) = (7)(.2) + (4)(.5) + (2)(.3) = 4$$

$$E(\Delta Y) = (-20)(.2) + (6)(.5) + (35)(.3) = 9.5$$

Computing the Standard Deviation for Investment Returns

		Investment	
P(ΔX,ΔY)	Economic condition	Bond Fund X	Aggressive Fund Y
0.2	Recession	+ $ 7	- $20
0.5	Stable Economy	+ 4	+ 6
0.3	Expanding Economy	+ 2	+ 35

$$\sigma_{\Delta X} = \sqrt{\mathrm{Var}(\Delta X)} = \sqrt{(7-4)^2(0.2)+(4-4)^2(0.5)+(2-4)^2(0.3)}$$
$$= \sqrt{3} \approx 1.73$$

$$\sigma_{\Delta Y} = \sqrt{\mathrm{Var}(\Delta Y)} = \sqrt{(-20-9.5)^2(0.2)+(6-9.5)^2(0.5)+(35-9.5)^2(0.3)}$$
$$= \sqrt{375.25} \approx 19.37$$

Covariance for Investment Returns

		Investment	
P(ΔX,ΔY)	Economic condition	Bond Fund X	Aggressive Fund Y
0.2	Recession	+ $ 7	- $20
0.5	Stable Economy	+ 4	+ 6
0.3	Expanding Economy	+ 2	+ 35

$$\sigma_{\Delta X \Delta Y} = \text{Cov}(\Delta X, \Delta Y) = (7-4)(-20-9.5)(.2) + (4-4)(6-9.5)(.5)$$
$$+ (2-4)(35-9.5)(.3)$$
$$= -33$$

Portfolio Example

Investment X:	$E(\Delta X) = 4$	$\sigma_{\Delta X} = 1.73$
Investment Y:	$E(\Delta Y) = 9.5$	$\sigma_{\Delta Y} = 19.32$
		$\sigma_{\Delta X \Delta Y} = -33$

Suppose 40% of the portfolio (W) is in Investment X and 60% is in Investment Y:

$$\boxed{E(\Delta W) = .4(4) + (.6)(9.5) = 7.3}$$

$$Var(\Delta W) = \sqrt{.4(1.73)^2 + .6(19.32)^2 + 2(.4)(.6)(-33)} = 14.47$$

The portfolio return and portfolio variability are between the values for investments X and Y considered individually

Interpreting the Results for Investment Returns

- The aggressive fund has a higher expected return, but much more risk

$$E(\Delta Y) = 9.5 \;>\; E(\Delta X) = 4$$

but

$$\sigma_{\Delta Y} = 19.32 \;>\; \sigma_{\Delta X} = 1.73$$

- The Covariance of -33 indicates that the two investments are negatively related and will vary in the opposite direction