# Lecture 4: Proofs for Expectation, Variance, and Covariance Formula

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# Discrete Random Variables: $X$ and $Y$

- Let $X$ and $Y$ be two discrete random variables.
- $X$ takes $n$ possible values: $\{x_1, \ldots, x_n\}$
- $Y$ takes $m$ possible values: $\{y_1, \ldots, y_m\}$.

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# Notations:

The **joint probability mass function** is given by

$$p_{ij}^{X,Y} = P\left(X = x_i, Y = y_j\right), \qquad i = 1, \ldots n; j = 1, \ldots, m.$$

The **marginal probability mass function of** $X$ is

$$p_i^X = P\left(X = x_i\right) = \sum_{j=1}^{m} p_{ij}^{X,Y}, \qquad i = 1, \ldots n,$$

and the **marginal probability mass function of** $Y$ is

$$p_j^Y = P\left(Y = y_j\right) = \sum_{i=1}^{n} p_{ij}^{X,Y}, \qquad j = 1, \ldots m.$$

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# Notations:

Table: Example of Joint Distribution of X and Y

| | $Y = y_1$ | $Y = y_2$ | $Y = y_3$ | Marg. prob. of $X$ |
|---|---|---|---|---|
| $X = x_1$ | $p^{X,Y}_{11}$ | $p^{X,Y}_{12}$ | $p^{X,Y}_{13}$ | $p^X_1$ |
| $X = x_2$ | $p^{X,Y}_{21}$ | $p^{X,Y}_{22}$ | $p^{X,Y}_{23}$ | $p^X_2$ |
| Marg. prob. of Y | $p^Y_1$ | $p^Y_2$ | $p^Y_3$ | 1.00 |

$$P\left(X = x_1\right) = \sum_{j=1}^{3} p^{X,Y}_{1j} = p^{X,Y}_{11} + p^{X,Y}_{12} + p^{X,Y}_{13} = p^X_1.$$

# Notations:

Table: Example of Joint Distribution of X and Y

| | Y = 30 | Y = 60 | Y = 100 | Marg. Prob. of X |
|---|---|---|---|---|
| X = 0 | 0.24 | 0.12 | 0.04 | 0.40 |
| X = 1 | 0.12 | 0.36 | 0.12 | 0.60 |
| Marg. Prob. of Y | 0.36 | 0.48 | 0.16 | 1.00 |

$$P(X = 0) = 0.24 + 0.12 + 0.04 = 0.40.$$

# Notations:

By definition,

$$E_X[X] = \sum_{i=1}^{n} x_i p_i^X, \quad E_Y[Y] = \sum_{i=1}^{m} y_i p_i^Y,$$

$$Var[X] = \sum_{i=1}^{n} (x_i - E_X[X])^2 p_i^X,$$

$$Cov[X, Y] = \sum_{i=1}^{n} \sum_{j=1}^{m} (x_i - E_X[X])(y_j - E_Y[Y]) p_{ij}^{X,Y}.$$

## Proposition

*If a and b are constants, then* $E(a+bX) = a + bE(X)$.

**Proof:**

$$
\begin{aligned}
&E(a+bX) \\
&\overset{\text{def}}{=} \sum_{i=1}^{n} (a+bx_i)p_i^X \\
&= (a+bx_1)p_1^X + .... + (a+bx_n)p_n^X \\
&= (ap_1^X + ... + ap_n^X) + (bx_1p_1^X + .... + bx_np_n^X) \\
&= a \times (p_1^X + ... + p_n^X) + b \times (x_1p_1^X + .... + x_np_n^X) \\
&= a\sum_{i=1}^{n} p_i^X + b\sum_{i=1}^{n} x_ip_i^X \\
&= a + bE(X).
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# Clicker Question 4-1

For any constant $a$, $b$, and for any function $g(x)$, which of the following is true?

A). $E[a + b \times g(X)] = a + b \times g(E[X])$.

B). $E[a + b \times g(X)] = a + b \times E[g(X)]$.

C). $E[a + b \times g(X)] = a + b \times g(E[X]) = a + b \times E[g(X)]$.

# Question 1 in Worksheet

Prove that, for any constant $a$, $b$, and for any function $g(x)$,

$$E[a + b \times g(X)] = a + b \times E[g(X)].$$

## Proposition

*$Cov\left(a_1 + b_1X, a_2 + b_2Y\right) = b_1b_2Cov\left(X, Y\right)$, where $a_1, a_2, b_1$, and $b_2$ are some constants.*

**Proof:**

$$
\begin{aligned}
&Cov(a_1 + b_1X, a_2 + b_2Y) \\
&\overset{\text{def}}{=} E[(a_1 + b_1X - E(a_1 + b_1X))(a_2 + b_2Y - E(a_2 + b_2Y))] \\
&= E[(a_1 - a_1 + b_1X - b_1E(X))(a_2 - a_2 + b_2Y - b_2E(Y))] \\
&= E[b_1b_2(X - E(X))(Y - E(Y))] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} b_1b_2(x_i - E(X))(y_j - E(Y)) \cdot p_{ij}^{X,Y} \\
&= b_1b_2\sum_{i=1}^{n}\sum_{j=1}^{m}[x_i - E(X)][y_j - E(Y)] \cdot p_{ij}^{X,Y} \\
&= b_1b_2Cov(X, Y).
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## Proposition

*$Cov(a_1 + b_1 X, a_2 + b_2 Y) = b_1 b_2 Cov(X, Y)$, where $a_1, a_2, b_1$, and $b_2$ are some constants.*

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& = E[b_1 b_2 (X - E(X))(Y - E(Y))] \\
& = \sum_{i=1}^{n} \sum_{j=1}^{m} b_1 b_2 (x_i - E(X))(y_j - E(Y)) \cdot p_{ij}^{X,Y} \\
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& = b_1 b_2 Cov(X, Y).
\end{aligned}
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# Clicker Question 4-2

For any constant $a$ and $b$, which of the following is true?

A). $Var[a + bX] = a + bVar[X]$.

B). $Var[a + bX] = bVar[X]$.

C). $Var[a + bX] = b^2 Var[X]$.

# Question 2 in Worksheet

Prove that, for any any constant $a$, $b$,

$$Var[a + bX] = b^2 Var[X].$$

## Proposition

*If X and Y are independent, then* $Cov(X,Y)=0$.

**Proof:**

$$
\begin{aligned}
& Cov(X,Y) \\
& \overset{\text{def}}{=} E[(X-E(X))(Y-E(Y))] \\
& = \sum_{i=1}^{n}\sum_{j=1}^{m}[x_i-E(X)][y_j-E(Y)]\cdot P(X=x_i,Y=y_j) \\
& = \sum_{i=1}^{n}\sum_{j=1}^{m}[x_i-E(X)][y_j-E(Y)]p_i^X p_j^Y \quad \text{(by independence)} \\
& = \sum_{i=1}^{n}\sum_{j=1}^{m}\{[x_i-E(X)]p_i^X\}\{[y_j-E(Y)]p_j^Y\}
\end{aligned}
$$

## Proposition

*If X and Y are independent, then Cov (X, Y) = 0.*

**Proof:**

$$Cov(X, Y)$$

$$\overset{\text{def}}{=} E[(X - E(X))(Y - E(Y))]$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} [x_i - E(X)][y_j - E(Y)] \cdot P(X = x_i, Y = y_j)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} [x_i - E(X)][y_j - E(Y)] p_i^X p_j^Y \quad \text{(by independence)}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \{[x_i - E(X)] p_i^X\} \{[y_j - E(Y)] p_j^Y\}$$

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# Proof continued

$$= \sum_{i=1}^{n}\sum_{j=1}^{m}\{[x_i - E(X)]p_i^X\}\{[y_j - E(Y)]p_j^Y\}$$

$$= \sum_{i=1}^{n}[x_i - E(X)]p_i^X \left\{\sum_{j=1}^{m}[y_j - E(Y)]p_j^Y\right\}$$

$$= \left\{\sum_{i=1}^{n} x_i p_i^X - \sum_{i=1}^{n} E(X)p_i^X\right\} \cdot \left\{\sum_{j=1}^{m} y_j p_j^Y - \sum_{j=1}^{m} E(Y)p_j^Y\right\}$$

$$= \left\{E(X) - E(X)\sum_{i=1}^{n} p_i^X\right\} \cdot \left\{E(Y) - E(Y)\sum_{j=1}^{m} p_j^Y\right\}$$

(by definition of $E(X)$ and $E(Y)$)

# Proof continued

$$= \sum_{i=1}^{n}\sum_{j=1}^{m}\{[x_i - E(X)]p_i^X\}\{[y_j - E(Y)]p_j^Y\}$$

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(by definition of $E(X)$ and $E(Y)$)

# Proof continued

$$= \left\{ E(X) - E(X) \sum_{i=1}^{n} p_i^X \right\} \cdot \left\{ E(Y) - E(Y) \sum_{j=1}^{m} p_j^Y \right\}$$

$$= \{E(X) - E(X) \cdot 1\} \cdot \{E(Y) - E(Y) \cdot 1\}$$

$$= 0 \cdot 0 = 0.$$

# Proof continued

$$= \left\{ E(X) - E(X) \sum_{i=1}^{n} p_i^X \right\} \cdot \left\{ E(Y) - E(Y) \sum_{j=1}^{m} p_j^Y \right\}$$
$$= \{E(X) - E(X) \cdot 1\} \cdot \{E(Y) - E(Y) \cdot 1\}$$
$$= 0 \cdot 0 = 0.$$

# Proof continued

$$= \left\{ E(X) - E(X) \sum_{i=1}^{n} p_i^X \right\} \cdot \left\{ E(Y) - E(Y) \sum_{j=1}^{m} p_j^Y \right\}$$

$$= \{E(X) - E(X) \cdot 1\} \cdot \{E(Y) - E(Y) \cdot 1\}$$

$$= 0 \cdot 0 = 0.$$

# Clicker Question 4-3

Suppose that $X$ and $Y$ are independent. For any function $g(x)$ and $h(y)$, which of the following is true?

A). $Cov\,(g(X), h(Y)) = 0$ always holds.

B). $Cov\,(g(X), h(Y)) = 0$ does not hold in some cases.

# Question 4 in Worksheet

Prove that, when $X$ and $Y$ are independent,

$$Cov\left(g(X), h(Y)\right) = 0$$

for any function $g(x)$ and $h(y)$.

## Proposition

$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y).$

**Proof:** Let $\tilde{X} \overset{\text{def}}{=} X - E(X)$ and $\tilde{Y} \overset{\text{def}}{=} Y - E(Y)$.

$$\begin{aligned}
&Var(X+Y) \\
&\overset{\text{def}}{=} E[(X+Y-E(X+Y))^2] \\
&= E[((X-E(X))+(Y-E(Y)))^2] \\
&= E[(\tilde{X}+\tilde{Y})^2] \\
&= E[\tilde{X}^2+\tilde{Y}^2+2\tilde{X}\tilde{Y}] \\
&= E[((X-E(X))^2]+E[(Y-E(Y))^2] \\
&\quad +2E[((X-E(X))(Y-E(Y))] \\
&= Var(X)+Var(Y)+2Cov(X,Y)
\end{aligned}$$

by definition of variance and covariance

## Proposition

$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y).$$

**Proof:** Let $\tilde{X} \stackrel{\text{def}}{=} X - E(X)$ and $\tilde{Y} \stackrel{\text{def}}{=} Y - E(Y)$.

$$
\begin{aligned}
& Var(X+Y) \\
& \stackrel{\text{def}}{=} E[(X+Y-E(X+Y))^2] \\
& = E[((X-E(X)) + (Y-E(Y)))^2] \\
& = E[(\tilde{X}+\tilde{Y})^2] \\
& = E[\tilde{X}^2 + \tilde{Y}^2 + 2\tilde{X}\tilde{Y}] \\
& = E[((X-E(X))^2] + E[(Y-E(Y))^2] \\
& \quad + 2E[((X-E(X))(Y-E(Y))] \\
& = Var(X) + Var(Y) + 2Cov(X,Y)
\end{aligned}
$$

by definition of variance and covariance

## Proposition

$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y).$$

**Proof:** Let $\tilde{X} \overset{\text{def}}{=} X - E(X)$ and $\tilde{Y} \overset{\text{def}}{=} Y - E(Y)$.

$$
\begin{aligned}
&Var(X+Y) \\
&\overset{\text{def}}{=} E[(X+Y-E(X+Y))^2] \\
&= E[((X-E(X)) + (Y-E(Y)))^2] \\
&= E[(\tilde{X}+\tilde{Y})^2] \\
&= E[\tilde{X}^2 + \tilde{Y}^2 + 2\tilde{X}\tilde{Y}] \\
&= E[((X-E(X))^2] + E[(Y-E(Y))^2] \\
&\quad + 2E[((X-E(X))(Y-E(Y))] \\
&= Var(X) + Var(Y) + 2Cov(X,Y)
\end{aligned}
$$

by definition of variance and covariance

**Proposition**

$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y).$

**Proof:** Let $\tilde{X} \overset{\text{def}}{=} X - E(X)$ and $\tilde{Y} \overset{\text{def}}{=} Y - E(Y)$.

$$
\begin{aligned}
& Var(X+Y) \\
& \overset{\text{def}}{=} E[(X+Y-E(X+Y))^2] \\
& = E[((X-E(X)) + (Y-E(Y)))^2] \\
& = E[(\tilde{X}+\tilde{Y})^2] \\
& = E[\tilde{X}^2 + \tilde{Y}^2 + 2\tilde{X}\tilde{Y}] \\
& = E[(X-E(X))^2] + E[(Y-E(Y))^2] \\
& \quad + 2E[(X-E(X))(Y-E(Y))] \\
& = Var(X) + Var(Y) + 2Cov(X,Y)
\end{aligned}
$$

by definition of variance and covariance

## Proposition

$$Var\,(X+Y) = Var\,(X) + Var\,(Y) + 2Cov\,(X,Y).$$

**Proof:** Let $\tilde{X} \stackrel{\text{def}}{=} X - E(X)$ and $\tilde{Y} \stackrel{\text{def}}{=} Y - E(Y)$.

$$\begin{aligned}
&Var(X+Y)\\
&\stackrel{\text{def}}{=} E[(X+Y-E(X+Y))^2]\\
&= E[((X-E(X)) + (Y-E(Y)))^2]\\
&= E[(\tilde{X}+\tilde{Y})^2]\\
&= E[\tilde{X}^2 + \tilde{Y}^2 + 2\tilde{X}\tilde{Y}]\\
&= E[((X-E(X))^2] + E[(Y-E(Y))^2]\\
&\quad + 2E[((X-E(X))(Y-E(Y))]\\
&= Var(X) + Var(Y) + 2Cov(X,Y)
\end{aligned}$$

by definition of variance and covariance

## Proposition

$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y).$$

**Proof:** Let $\tilde{X} \overset{\text{def}}{=} X - E(X)$ and $\tilde{Y} \overset{\text{def}}{=} Y - E(Y)$.

$$
\begin{aligned}
& Var(X+Y) \\
& \overset{\text{def}}{=} E[(X+Y-E(X+Y))^2] \\
& = E[((X-E(X)) + (Y-E(Y)))^2] \\
& = E[(\tilde{X}+\tilde{Y})^2] \\
& = E[\tilde{X}^2 + \tilde{Y}^2 + 2\tilde{X}\tilde{Y}] \\
& = E[((X-E(X))^2] + E[(Y-E(Y))^2] \\
& \quad + 2E[((X-E(X))(Y-E(Y))] \\
& = Var(X) + Var(Y) + 2Cov(X,Y)
\end{aligned}
$$

by definition of variance and covariance

## Proposition

$Var\,(X+Y) = Var\,(X) + Var\,(Y) + 2Cov\,(X,Y).$

**Proof:** Let $\tilde{X} \stackrel{\text{def}}{=} X - E(X)$ and $\tilde{Y} \stackrel{\text{def}}{=} Y - E(Y)$.

$$
\begin{aligned}
& Var(X+Y) \\
& \stackrel{\text{def}}{=} E[(X+Y-E(X+Y))^2] \\
& = E[((X-E(X)))+(Y-E(Y)))^2] \\
& = E[(\tilde{X}+\tilde{Y})^2] \\
& = E[\tilde{X}^2+\tilde{Y}^2+2\tilde{X}\tilde{Y}] \\
& = E[((X-E(X))^2]+E[(Y-E(Y))^2] \\
& \quad +2E[((X-E(X))(Y-E(Y))] \\
& = Var(X)+Var(Y)+2Cov(X,Y)
\end{aligned}
$$

by definition of variance and covariance

# Clicker Question 4-4

For any function $g(x)$ and $h(y)$, which of the following is true?

A). $Var(g(X) + h(Y)) = Var(g(X)) + Var(h(Y))$

B). $Var(g(X) + h(Y))$
$= Var(g(X)) + Var(h(Y)) + 2Cov(g(X), h(Y))$

C). $Var(g(X) + h(Y)) = g(Var(X)) + h(Var(Y))$

# Question 5 in Worksheet

Prove that, for any constant $a$ and $b$,

$$Var[aX + bY] = a^2 Var[X] + b^2 Var[Y] + 2abCov[X, Y].$$

# Bernoulli random variable

- The probability mass function:

$$X = \begin{cases} 0 & \text{with probability } \quad 1-p \\ 1 & \text{with probability } p \end{cases}$$

- What is $E[X]$ and $Var[X]$?

## Proposition

*If $X$ is a Bernoulli random variable with $P(X = 1) = p$, then $E[X] = p$.*

**Proof:**

$$E(X) \overset{\text{def}}{=} \sum_{x=0,1} x p^x (1-p)^{1-x}$$
$$= (0)(1-p) + (1)(p)$$
$$= p$$

## Proposition

*If $X$ is a Bernoulli random variable with $P(X = 1) = p$, then $E[X] = p$.*

**Proof:**

$$E(X) \overset{\text{def}}{=} \sum_{x=0,1} x p^x (1-p)^{1-x}$$

$$= (0)(1-p) + (1)(p)$$

$$= p$$

## Proposition

*If $X$ is a Bernoulli random variable with $P(X = 1) = p$, then $E[X] = p$.*

**Proof:**

$$
\begin{aligned}
E(X) &\overset{\text{def}}{=} \sum_{x=0,1} x p^x (1-p)^{1-x} \\
&= (0)(1-p) + (1)(p) \\
&= p
\end{aligned}
$$

# Clicker Question 4-6

Suppose that $X_1$ and $X_2$ are two Bernoulli random variables that are stochastically independent with $P(X_1 = 1) = P(X_2 = 1) = p$. Which of the following is true?

A). $E[X_1 + X_2] = p$.

B). $E[X_1 + X_2] = 2p$.

C). $E[X_1 + X_2] = p/2$.

# Question 7 in Worksheet

Prove that, if $X$ and $Y$ are two Bernoulli random variables that are stochastically independent with $P(X = 1) = P(Y = 1) = p$, then

$$E[X + Y] = 2p.$$

## Proposition

*If $X$ is a Bernoulli random variable with $P(X = 1) = p$, then $V[X] = p(1 - p)$.*

**Proof:**

$$\begin{aligned}
\text{Var}(X) &\stackrel{\text{def}}{=} \sum_{x=0,1} (x - p)^2 p^x (1 - p)^{1-x} \\
&= (0 - p)^2 (1 - p) + (1 - p)^2 p \\
&= p^2 (1 - p) + (1 - p)^2 p \\
&= (p + (1 - p)) \times p(1 - p) \\
&= p(1 - p).
\end{aligned}$$

## Proposition

*If X is a Bernoulli random variable with $P(X = 1) = p$, then $V[X] = p(1 - p)$.*

**Proof:**

$$\mathrm{Var}(X) \overset{\text{def}}{=} \sum_{x=0,1} (x - p)^2 p^x (1 - p)^{1-x}$$
$$= (0 - p)^2(1 - p) + (1 - p)^2 p$$
$$= p^2(1 - p) + (1 - p)^2 p$$
$$= (p + (1 - p)) \times p(1 - p)$$
$$= p(1 - p).$$

## Proposition

*If $X$ is a Bernoulli random variable with $P(X = 1) = p$, then $V[X] = p(1 - p)$.*

**Proof:**

$$
\begin{aligned}
\text{Var}(X) &\overset{\text{def}}{=} \sum_{x=0,1} (x - p)^2 p^x (1 - p)^{1-x} \\
&= (0 - p)^2 (1 - p) + (1 - p)^2 p \\
&= p^2 (1 - p) + (1 - p)^2 p \\
&= (p + (1 - p)) \times p(1 - p) \\
&= p(1 - p).
\end{aligned}
$$

## Proposition

*If $X$ is a Bernoulli random variable with $P(X = 1) = p$, then $V[X] = p(1 - p)$.*

**Proof:**

$$\begin{aligned} \text{Var}(X) &\stackrel{\text{def}}{=} \sum_{x=0,1} (x-p)^2 p^x (1-p)^{1-x} \\ &= (0-p)^2(1-p) + (1-p)^2 p \\ &= p^2(1-p) + (1-p)^2 p \\ &= (p + (1-p)) \times p(1-p) \\ &= p(1-p). \end{aligned}$$

## Proposition

*If $X$ is a Bernoulli random variable with $P(X = 1) = p$, then $V[X] = p(1 - p)$.*

**Proof:**

$$\begin{aligned}
\mathrm{Var}(X) &\overset{\text{def}}{=} \sum_{x=0,1} (x - p)^2 p^x (1 - p)^{1-x} \\
&= (0 - p)^2(1 - p) + (1 - p)^2 p \\
&= p^2(1 - p) + (1 - p)^2 p \\
&= (p + (1 - p)) \times p(1 - p) \\
&= p(1 - p).
\end{aligned}$$

# Clicker Question 4-7

Suppose that $X_1$ and $X_2$ are two Bernoulli random variables that are stochastically independent with $P(X_i = 1)$ for $i = 1, 2$. Which of the following is true?

A). $Var[(X_1 + X_2)/2] = p(1 - p)$.

B). $Var[(X_1 + X_2)/2] = p(1 - p)/2$.

C). $Var[(X_1 + X_2)/2] = p(1 - p)/4$.

# Question 9 in Worksheet

Prove that, if $X$ and $Y$ are two Bernoulli random variables that are stochastically independent with $P(X = 1) = P(Y = 1) = p$, then

$$Var\left[\frac{X+Y}{2}\right] = \frac{p(1-p)}{2}.$$

## Proposition

*Suppose that $X_i$ for $i = 1, ..., n$ are $n$ Bernoulli random variables that are stochastically independent to each other with $P(X_i = 1)$ for $i = 1, ..., n$. Let $\bar{X} := \frac{1}{n}\sum_{i=1}^{n} X_i$. Then, (1) $E[\bar{X}] = p$ and (2) $Var[\bar{X}] = \frac{p(1-p)}{n}$.*

**Proof for (1):**

$$E[\bar{X}] = \frac{1}{n}E[X_1 + X_2 + ... + X_n]$$

$$= \frac{1}{n}\{E[X_1] + E[X_2] + ... + E[X_n]\}$$

$$= \frac{p + p + ... + p}{n}$$

$$= \frac{n \times p}{n} = p.$$

## Proposition

*Suppose that $X_i$ for $i = 1, ..., n$ are $n$ Bernoulli random variables that are stochastically independent to each other with $P(X_i = 1)$ for $i = 1, ..., n$. Let $\bar{X} := \frac{1}{n}\sum_{i=1}^{n} X_i$. Then, (1) $E[\bar{X}] = p$ and (2) $Var[\bar{X}] = \frac{p(1-p)}{n}$.*

**Proof for (1):**

$$E[\bar{X}] = \frac{1}{n}E[X_1 + X_2 + ... + X_n]$$
$$= \frac{1}{n}\{E[X_1] + E[X_2] + ... + E[X_n]\}$$
$$= \frac{p + p + ... + p}{n}$$
$$= \frac{n \times p}{n} = p.$$

## Proposition

*Suppose that $X_i$ for $i = 1, ..., n$ are $n$ Bernoulli random variables that are stochastically independent to each other with $P(X_i = 1)$ for $i = 1, ..., n$. Let $\bar{X} := \frac{1}{n}\sum_{i=1}^{n} X_i$. Then, (1) $E[\bar{X}] = p$ and (2) $Var[\bar{X}] = \frac{p(1-p)}{n}$.*

**Proof for (1):**

$$\begin{aligned} E[\bar{X}] &= \frac{1}{n}E[X_1 + X_2 + ... + X_n] \\ &= \frac{1}{n}\{E[X_1] + E[X_2] + ... + E[X_n]\} \\ &= \frac{p + p + ... + p}{n} \\ &= \frac{n \times p}{n} = p. \end{aligned}$$

## Proposition

*Suppose that $X_i$ for $i = 1, ..., n$ are $n$ Bernoulli random variables that are stochastically independent to each other with $P(X_i = 1)$ for $i = 1, ..., n$. Let $\bar{X} := \frac{1}{n}\sum_{i=1}^{n} X_i$. Then, (1) $E[\bar{X}] = p$ and (2) $Var[\bar{X}] = \frac{p(1-p)}{n}$.*

**Proof for (1):**

$$E[\bar{X}] = \frac{1}{n}E[X_1 + X_2 + ... + X_n]$$
$$= \frac{1}{n}\{E[X_1] + E[X_2] + ... + E[X_n]\}$$
$$= \frac{p + p + ... + p}{n}$$
$$= \frac{n \times p}{n} = p.$$

## Proof for (2) $Var[\bar{X}] = \frac{p(1-p)}{n}$:

$$Var[\bar{X}] = Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right]$$

$$= \left(\frac{1}{n}\right)^2 Var[X_1 + X_2 + ... + X_n]$$

$$= \frac{1}{n^2}\{Var[X_1] + Var[X_2] + ... + Var[X_n]$$
$$+2Cov(X_1, X_2) + 2Cov(X_1, X_3) + ... + 2Cov(X_{n-1}, X_n)\}$$

$$= \frac{1}{n^2}\{p(1-p) + p(1-p) + ... + p(1-p) + 0\}$$

$$= \frac{n \times p(1-p)}{n^2}$$

$$= \frac{p(1-p)}{n}.$$

## Proof for (2) $Var[\bar{X}] = \frac{p(1-p)}{n}$:

$$Var[\bar{X}] = Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right]$$

$$= \left(\frac{1}{n}\right)^2 Var[X_1 + X_2 + ... + X_n]$$

$$= \frac{1}{n^2}\{Var[X_1] + Var[X_2] + ... + Var[X_n]$$

$$+2Cov(X_1, X_2) + 2Cov(X_1, X_3) + ... + 2Cov(X_{n-1}, X_n)\}$$

$$= \frac{1}{n^2}\{p(1-p) + p(1-p) + ... + p(1-p) + 0\}$$

$$= \frac{n \times p(1-p)}{n^2}$$

$$= \frac{p(1-p)}{n}.$$

## Proof for (2) $Var[\bar{X}] = \frac{p(1-p)}{n}$:

$$
\begin{aligned}
Var[\bar{X}] &= Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \\
&= \left(\frac{1}{n}\right)^2 Var[X_1 + X_2 + ... + X_n] \\
&= \frac{1}{n^2}\{Var[X_1] + Var[X_2] + ... + Var[X_n] \\
&\quad + 2Cov(X_1, X_2) + 2Cov(X_1, X_3) + ... + 2Cov(X_{n-1}, X_n)\} \\
&= \frac{1}{n^2}\{p(1-p) + p(1-p) + ... + p(1-p) + 0\} \\
&= \frac{n \times p(1-p)}{n^2} \\
&= \frac{p(1-p)}{n}.
\end{aligned}
$$

**Proof for (2)** $Var[\bar{X}] = \frac{p(1-p)}{n}$**:**

$$Var[\bar{X}] = Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right]$$

$$= \left(\frac{1}{n}\right)^2 Var[X_1 + X_2 + ... + X_n]$$

$$= \frac{1}{n^2}\{Var[X_1] + Var[X_2] + ... + Var[X_n]$$

$$+2Cov(X_1, X_2) + 2Cov(X_1, X_3) + ... + 2Cov(X_{n-1}, X_n)\}$$

$$= \frac{1}{n^2}\{p(1-p) + p(1-p) + ... + p(1-p) + 0\}$$

$$= \frac{n \times p(1-p)}{n^2}$$

$$= \frac{p(1-p)}{n}.$$

**Proof for (2)** $Var[\bar{X}] = \frac{p(1-p)}{n}$**:**

$$
\begin{aligned}
Var[\bar{X}] &= Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \\
&= \left(\frac{1}{n}\right)^2 Var[X_1 + X_2 + ... + X_n] \\
&= \frac{1}{n^2}\{Var[X_1] + Var[X_2] + ... + Var[X_n] \\
&\quad + 2Cov(X_1, X_2) + 2Cov(X_1, X_3) + ... + 2Cov(X_{n-1}, X_n)\} \\
&= \frac{1}{n^2}\{p(1-p) + p(1-p) + ... + p(1-p) + 0\} \\
&= \frac{n \times p(1-p)}{n^2} \\
&= \frac{p(1-p)}{n}.
\end{aligned}
$$

**Proof for (2)** $Var[\bar{X}] = \frac{p(1-p)}{n}$**:**

$$
\begin{aligned}
Var[\bar{X}] &= Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \\
&= \left(\frac{1}{n}\right)^2 Var[X_1 + X_2 + ... + X_n] \\
&= \frac{1}{n^2}\{Var[X_1] + Var[X_2] + ... + Var[X_n] \\
&\quad +2Cov(X_1, X_2) + 2Cov(X_1, X_3) + ... + 2Cov(X_{n-1}, X_n)\} \\
&= \frac{1}{n^2}\{p(1-p) + p(1-p) + ... + p(1-p) + 0\} \\
&= \frac{n \times p(1-p)}{n^2} \\
&= \frac{p(1-p)}{n}.
\end{aligned}
$$