Econ 325: Introduction to Empirical Economics

Lecture 5

Continuous Random Variables and Probability Distributions

5.1 Continuous Probability Distributions

- A continuous random variable is a variable that can assume any value in an interval
  - thickness of an item
  - time required to complete a task
  - height, in inches
- These can potentially take on any value, depending only on the ability to measure accurately.

Cumulative Distribution Function

- The cumulative distribution function, F(x), for a continuous random variable X expresses the probability that X does not exceed the value of x

$$F_X(x) = P(X \leq x)$$

- Let a and b be two possible values of X, with a < b. The probability that X lies between a and b is

$$P(a < X < b) = F_X(b) - F_X(a)$$

4.3

Definition: Probability density function

The probability density function (pdf) of a continuous random variable X is a function that satisfies the following properties:

1). $\mathbf{f}_X(\boldsymbol{x}) \geq \mathbf{0}$

2). $\int_{\boldsymbol{x} \in S_X} \mathbf{f}_X(\boldsymbol{x}) \boldsymbol{dx} = 1$

3). $\boldsymbol{P}(\boldsymbol{a} < \boldsymbol{X} < \boldsymbol{b}) = \int_{\boldsymbol{a}}^{\boldsymbol{b}} \mathbf{f}_X(\boldsymbol{x}) \boldsymbol{dx}$

Probability Density Function

(continued)

The cumulative distribution function (cdf) can be obtained from integrating the probability density function:

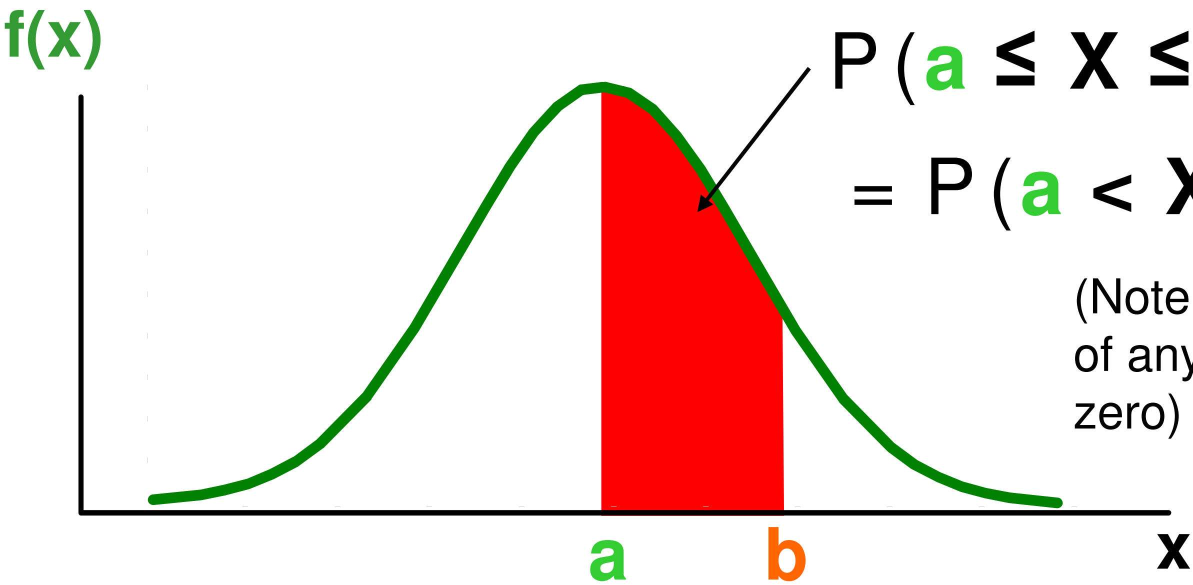
$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

The probability density function (pdf) can be obtained from differentiating the cumulative distribution function (cdf):

$$\frac{dF_X(x)}{dx} = f_X(x)$$

Probability as an Area

Shaded area under the curve is the probability that X is between a and b



$$P(a \leq X \leq b)$$

$$= P(a < X < b)$$

(Note that the probability of any individual value is zero)

Probability as an area under pdf

- Mathematically,

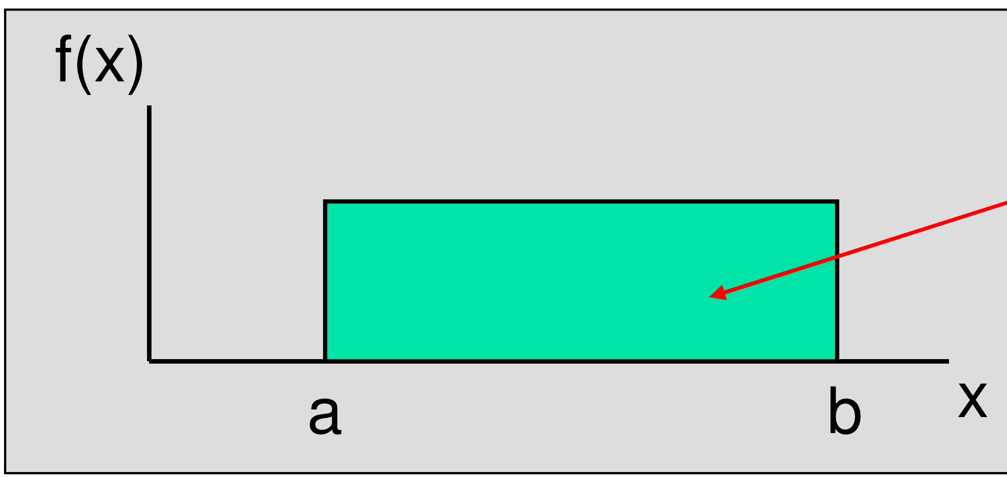
$$P(a < X < b) = \int_a^b f_X(t)dt$$
$$= F_X(b) - F_X(a)$$

- Also,

$$P(a < X < b) = P(X < b)\text{-}P(X < a)$$
$$= F_X(b) - F_X(a)$$

The Uniform Distribution

- The uniform distribution is a probability distribution that has equal probabilities for all possible outcomes of the random variable



Total area under the uniform probability density function is 1.0

The probability density function of a uniform random variable

(continued)

The probability density function of a uniform random variable:

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

When X is uniformly distributed on [a,b] , we write

$$\boldsymbol{X \sim U[a,b]}$$

Clicker Question 5-1

What is the cumulative distribution function of a random variable $X \sim U[a, b]$?

A) $F_X(x) = \frac{x}{b - a}$

B) $F_X(x) = \frac{x - a}{b - a}$

C) $F_X(x) = \frac{x - b}{b - a}$

$$\text{Hint}: F_X(x) = \int_a^x f_X(t)dt$$

Properties of the Uniform Distribution

- The mean of a uniform distribution is

$$\mu = \frac{a+b}{2}$$

- The variance is

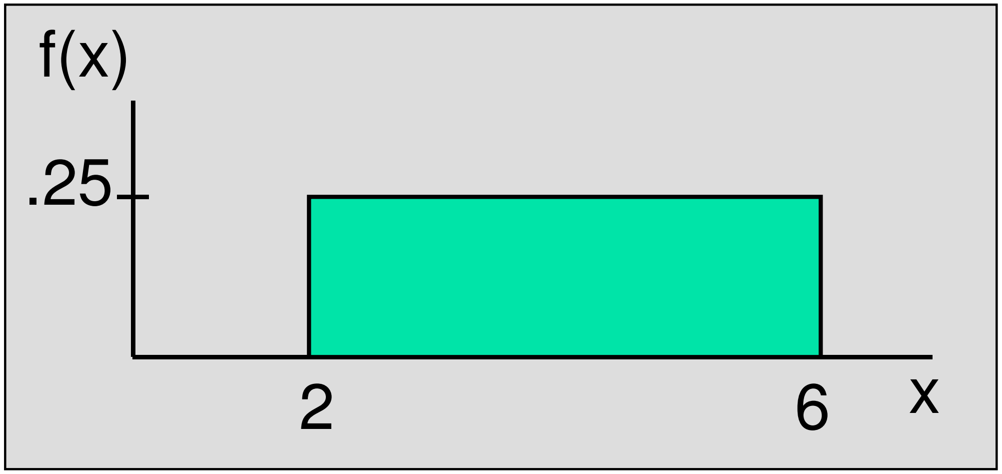
$$\sigma^2 = \frac{(b-a)^2}{12}$$

Example

uniform distribution over the range [2,6], i.e.,

$$X \sim U[2, 6]$$

$$f(x) = \frac{1}{6 - 2} = .25 \quad \text{for} \quad 2 \le x \le 6$$



$$\mu = \frac{a+b}{2} = \frac{2+6}{2} = 4$$

$$\sigma^2 = \frac{(b-a)^2}{12} = \frac{(6-2)^2}{12} = 1.333$$

Clicker Question 5-2

Suppose that $X \sim U[2, 6]$. What is P(3<X<5)?

A) 1/3

B) 1/2

C) 1/4

5.2

Expectations for Continuous Random Variables

- The mean of X, denoted μ_X, is defined as the expected value of X

$$\mu_X = E(X)$$

- The variance of X, denoted σ_X^2, is defined as the expectation of the squared deviation, $(X - \mu_X)^2$, of a random variable from its mean

$$\sigma_X^2 = E[(X - \mu_X)^2]$$

Linear Functions of Variables

- Let $W = a + bX$, where X has mean μ_X and variance σ_X^2, and a and b are constants
- Then the mean of W is

$$\mu_W = E(a + bX) = a + b\mu_X$$

- the variance is

$$\sigma_W^2 = Var(a + bX) = b^2\sigma_X^2$$

- the standard deviation of W is

$$\sigma_W = |b|\sigma_X$$

Linear Functions of Variables

(continued)

- An important special case of the previous results is the standardized random variable

$$Z = \frac{X - \mu_X}{\sigma_X}$$

Clicker Question 5-3

- What is the mean and the variance of the standardize random variable $Z = \frac{X - \mu_X}{\sigma_X}$?

A). E[Z]=0 and Var[Z]=0

B). E[Z]=1 and Var[Z]=1

C). E[Z]=1 and Var[Z]=0

D). E[Z]=0 and Var[Z]=1

The Normal Distribution

(continued)

- The normal distribution closely approximates the probability distributions of a wide range of random variables in empirical applications.

- Distributions of sample means approach a normal distribution given a "large" sample size (Central Limit Theorem)

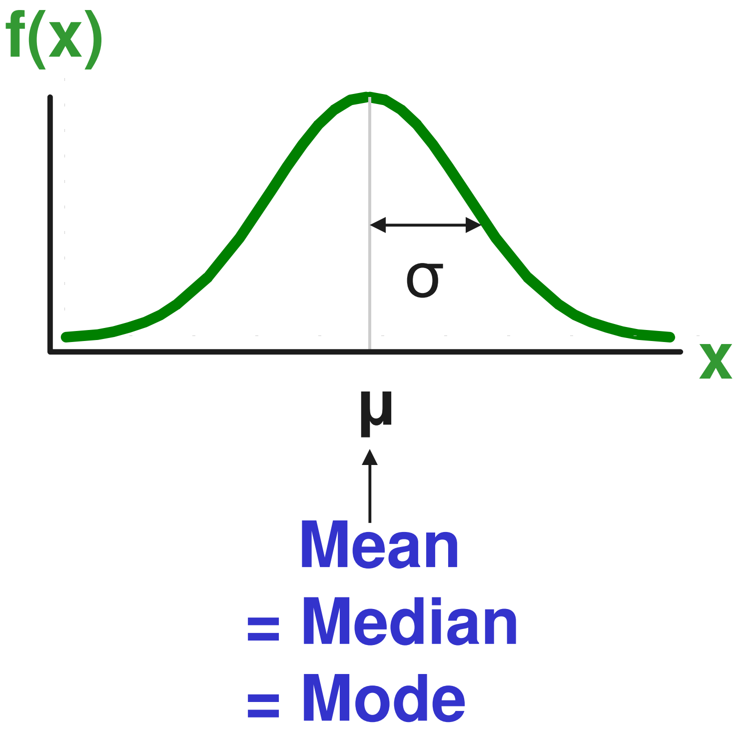
The Normal Distribution

(continued)

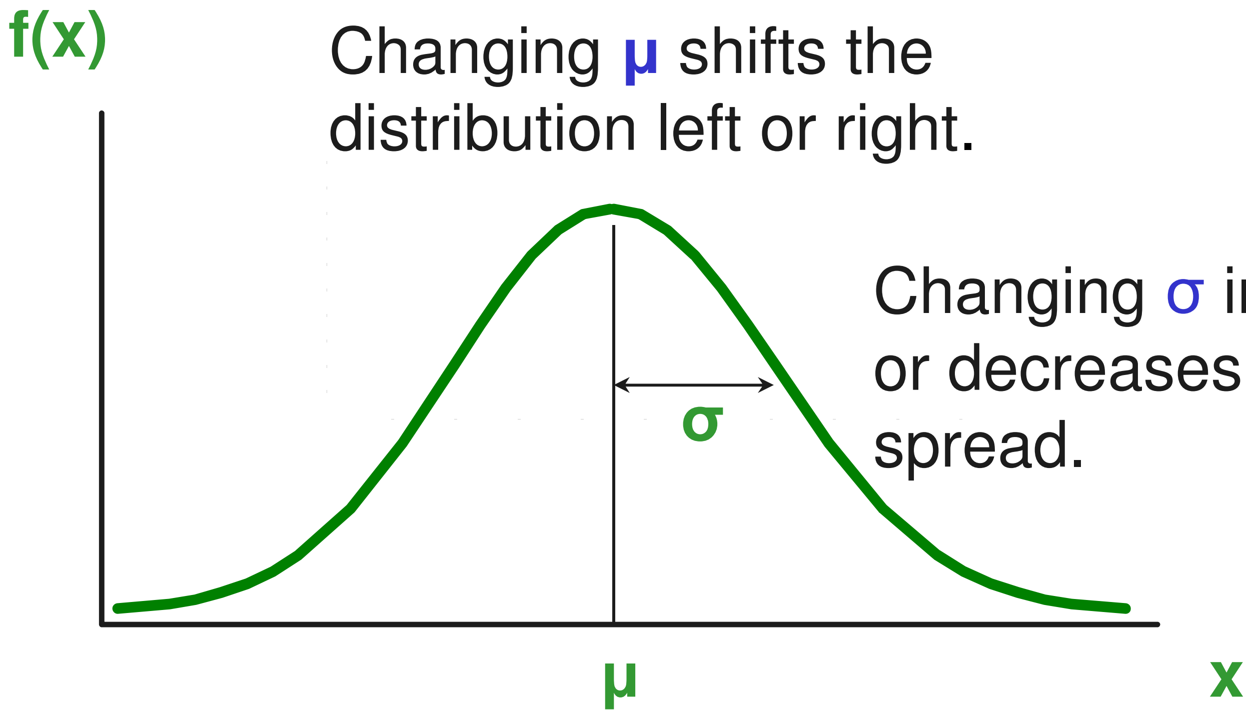
- Bell Shaped
- Symmetrical
- Mean, Median and Mode are Equal

The mean, μ, determines *location*.

The standard deviation, σ, determines the *spread*.



The Normal Distribution Shape



Changing μ shifts the distribution left or right.

Changing σ increases or decreases the spread.

Given the mean μ and variance σ we define the normal distribution using the notation

$$X \sim N(\mu, \sigma^2)$$

The Normal Probability Density Function

- The normal probability density function is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

Where e = the mathematical constant approximated by 2.71828

π = the mathematical constant approximated by 3.14159

μ = the population mean

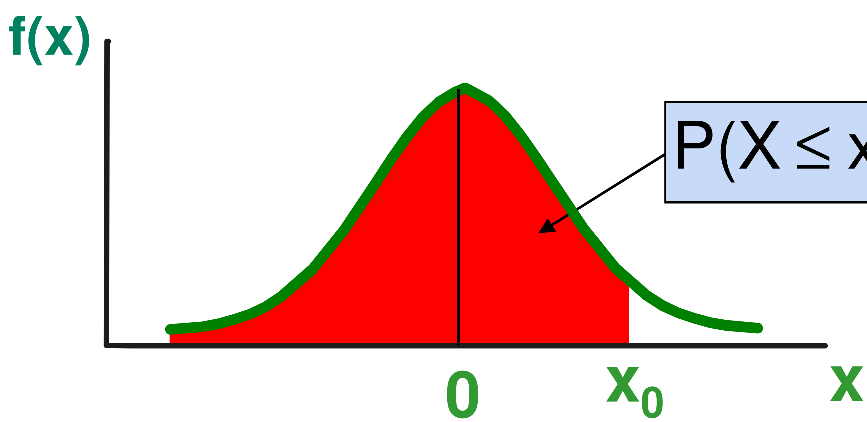
σ = the population standard deviation

x = any value of the continuous variable, $-\infty < x < \infty$

Cumulative Distribution

- For a normal random variable X with mean μ and variance σ^2 , i.e., $X \sim N(\mu, \sigma^2)$, the cumulative distribution function is

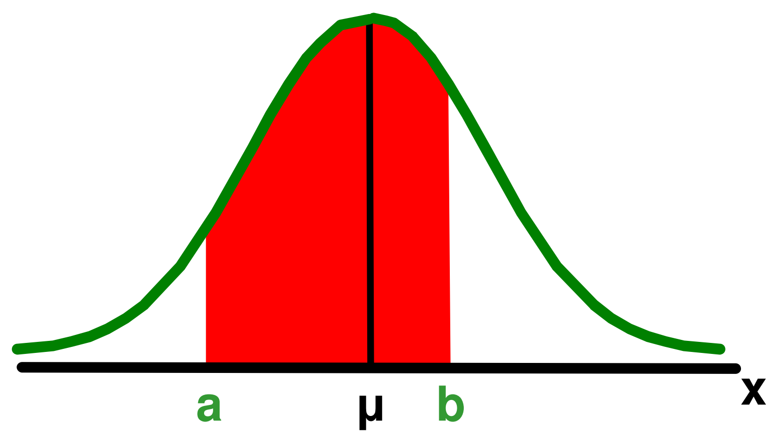
$$F(x_0) = P(X \le x_0)$$



Finding Normal Probabilities

The probability for a range of values is measured by the area under the curve

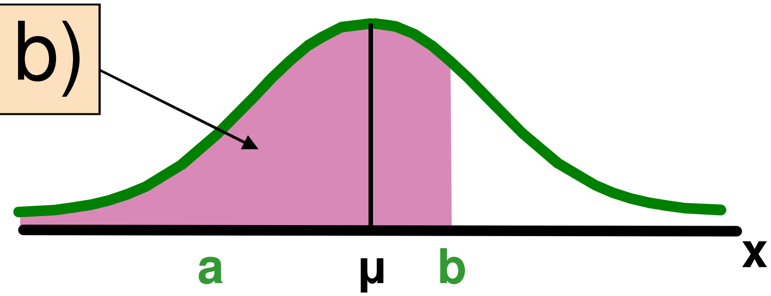
$$P(a < X < b) = F(b) - F(a)$$



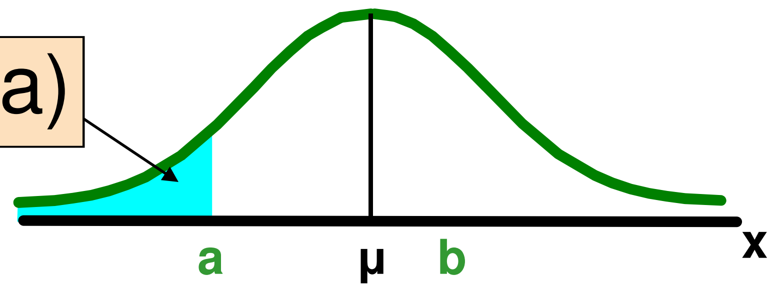
Finding Normal Probabilities

(continued)

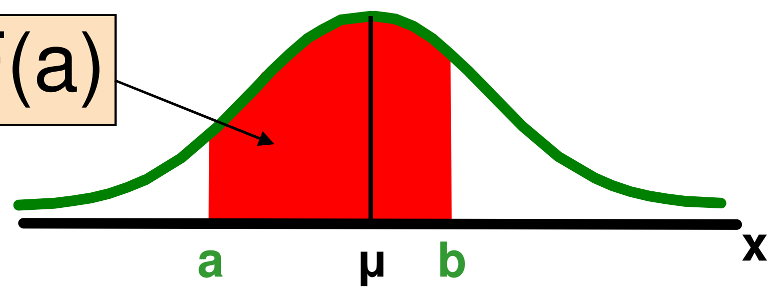
$$F(b) = P(X < b)$$



$$F(a) = P(X < a)$$



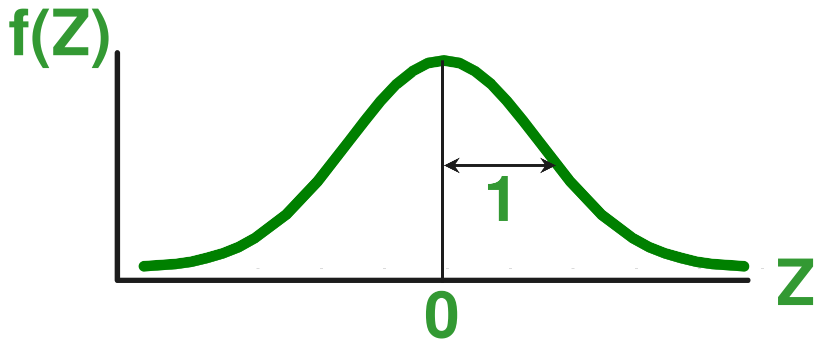
$$P(a < X < b) = F(b) - F(a)$$



The Standardized Normal

- Any normal distribution (with any mean and variance combination) can be transformed into the standardized normal distribution (Z), with mean 0 and variance 1

$$Z \sim N(0,1)$$



- Need to transform X units into Z units by subtracting the mean of X and dividing by its standard deviation

$$Z = \frac{X - \mu}{\sigma}$$

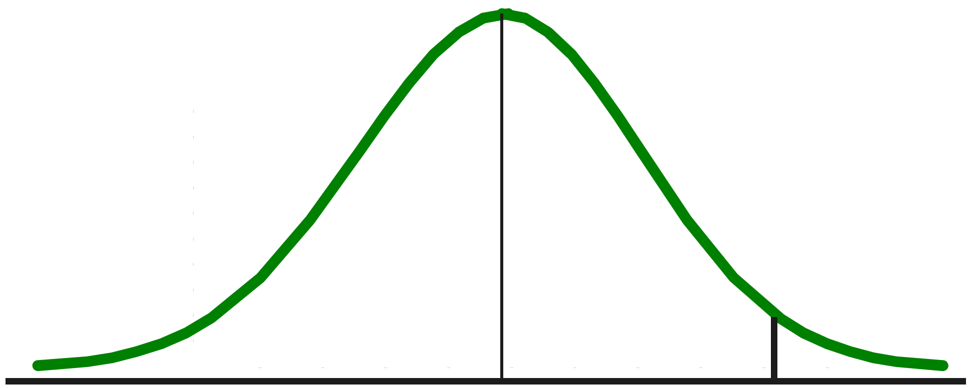
Example

- If X is distributed normally with mean of 100 and standard deviation of 50, the Z value for X = 200 is

$$Z = \frac{X - \mu}{\sigma} = \frac{200 - 100}{50} = 2.0$$

- This says that X = 200 is two standard deviations (2 increments of 50 units) above the mean of 100.

Comparing X and Z units



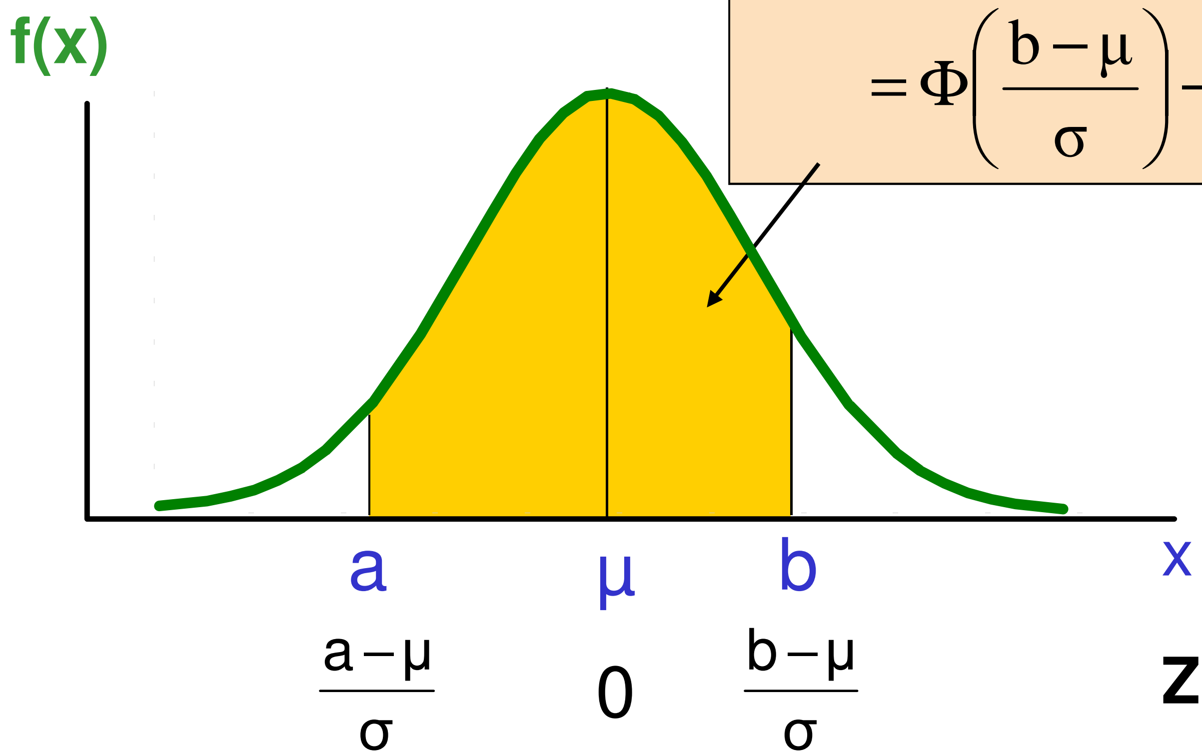
100	200	X	(μ = 100, σ = 50)
0	2.0	Z	(μ = 0 , σ = 1)

Note that the distribution is the same, only the scale has changed. We can express the problem in original units (X) or in standardized units (Z)

Finding Normal Probabilities

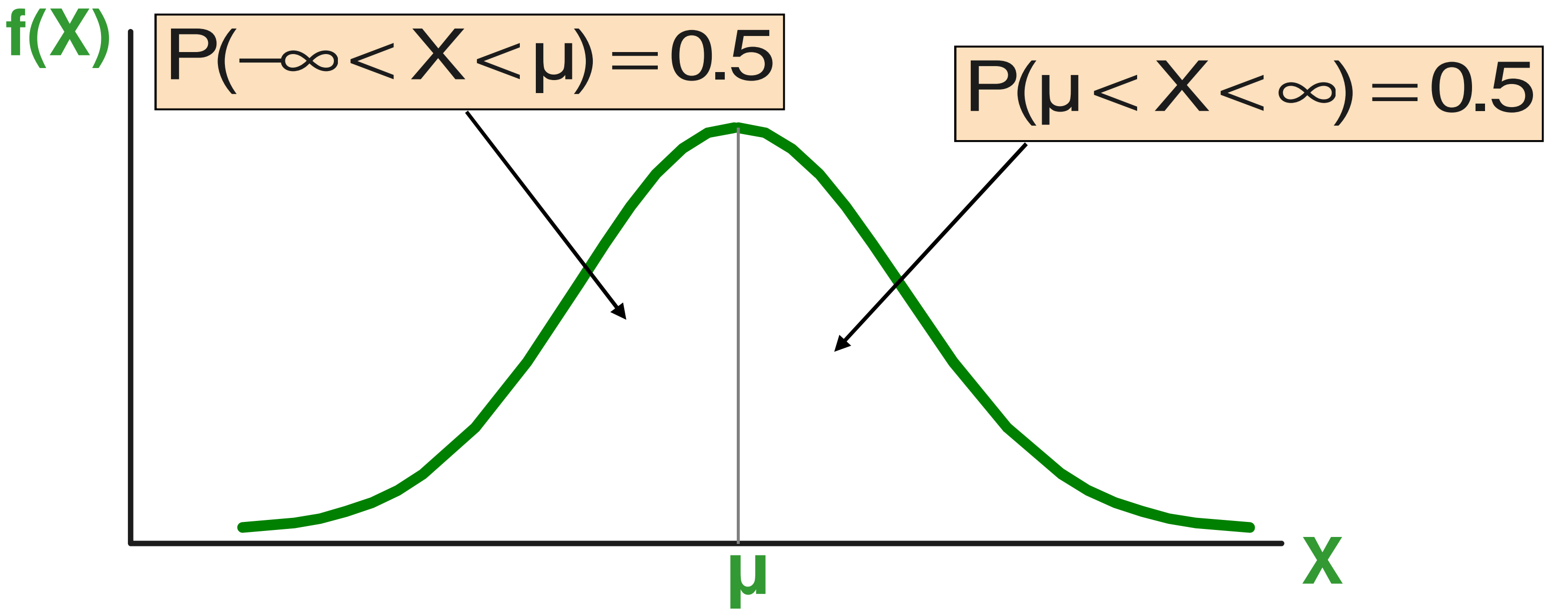
$$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$



Probability as Area Under the Curve

The total area under the curve is 1.0, and the curve is symmetric, so half is above the mean, half is below



$$P(-\infty < X < \infty) = 1.0$$

Appendix Table 1

- The Standardized Normal table in the textbook (Appendix Table 1) shows values of the cumulative normal distribution function

- For a given Z-value a, the table shows Φ(a)
  (the area under the curve from negative infinity to a)

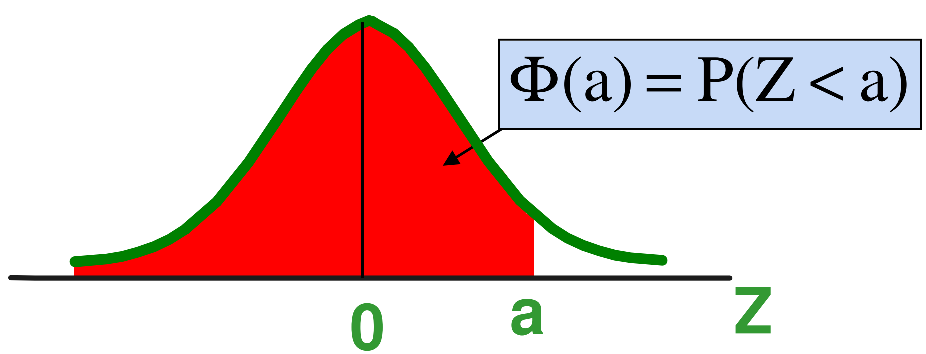
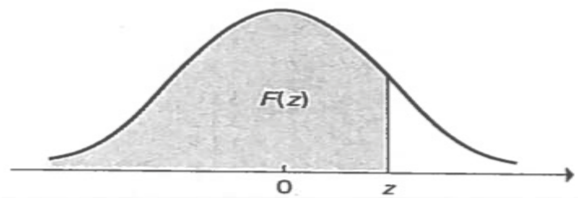


Table 1 Cumulative Distribution Function of the Standard Normal Distribution



z	F(z)	z	F(z)	z	F(z)	z	F(z)	z	F(z)	z
.00	.5000									
.01	.5040	.31	.6217	.61	.7291	.91	.8186	1.21	.8869	1.51
.02	.5080	.32	.6255	.62	.7324	.92	.8212	1.22	.8888	1.52
.03	.5120	.33	.6293	.63	.7357	.93	.8238	1.23	.8907	1.53
.04	.5160	.34	.6331	.64	.7389	.94	.8264	1.24	.8925	1.54
.05	.5199	.35	.6368	.65	.7422	.95	.8289	1.25	.8944	1.55
.06	.5239	.36	.6406	.66	.7454	.96	.8315	1.26	.8962	1.56
.07	.5279	.37	.6443	.67	.7486	.97	.8340	1.27	.8980	1.57
.08	.5319	.38	.6480	.68	.7517	.98	.8365	1.28	.8997	1.58
.09	.5359	.39	.6517	.69	.7549	.99	.8389	1.29	.9015	1.59
.10	.5398	.40	.6554	.70	.7580	1.00	.8413	1.30	.9032	1.60
.11	.5438	.41	.6591	.71	.7611	1.01	.8438	1.31	.9049	1.61
.12	.5478	.42	.6628	.72	.7642	1.02	.8461	1.32	.9066	1.62
.13	.5517	.43	.6664	.73	.7673	1.03	.8485	1.33	.9082	1.63
.14	.5557	.44	.6700	.74	.7704	1.04	.8508	1.34	.9099	1.64
.15	.5596	.45	.6736	.75	.7734	1.05	.8531	1.35	.9115	1.65
.16	.5636	.46	.6772	.76	.7764	1.06	.8554	1.36	.9131	1.66
.17	.5675	.47	.6803	.77	.7794	1.07	.8577	1.37	.9147	1.67
.18	.5714	.48	.6844	.78	.7823	1.08	.8599	1.38	.9162	1.68
.19	.5753	.49	.6879	.79	.7852	1.09	.8621	1.39	.9177	1.69
.20	.5793	.50	.6915	.80	.7881	1.10	.8643	1.40	.9192	1.70
.21	.5832	.51	.6950	.81	.7910	1.11	.8665	1.41	.9207	1.71
.22	.5871	.52	.6985	.82	.7939	1.12	.8686	1.42	.9222	1.72
.23	.5910	.53	.7019	.83	.7967	1.13	.8708	1.43	.9236	1.73
.24	.5948	.54	.7054	.84	.7995	1.14	.8729	1.44	.9251	1.74
.25	.5987	.55	.7088	.85	.8023	1.15	.8749	1.45	.9265	1.75
.26	.6026	.56	.7123	.86	.8051	1.16	.8770	1.46	.9279	1.76
.27	.6064	.57	.7157	.87	.8078	1.17	.8790	1.47	.9292	1.77
.28	.6103	.58	.7190	.88	.8106	1.18	.8810	1.48	.9306	1.78
.29	.6141	.59	.7224	.89	.8133	1.19	.8830	1.49	.9319	1.79
.30	.6179	.60	.7257	.90	.8159	1.20	.8849	1.50	.9332	1.80

Table 1 Cumulative Distribution Function of the Standard Normal Distribution Continue

z	F(z)	z	F(z)	z	F(z)	z	F(z)	z	F(z)
1.81	.9649	2.21	.9864	2.61	.9955	3.01	.9987	3.41	.9997
1.82	.9656	2.22	.9868	2.62	.9956	3.02	.9987	3.42	.9997
1.83	.9664	2.23	.9871	2.63	.9957	3.03	.9988	3.43	.9997
1.84	.9671	2.24	.9875	2.64	.9959	3.04	.9988	3.44	.9997
1.85	.9678	2.25	.9878	2.65	.9960	3.05	.9989	3.45	.9997
1.86	.9686	2.26	.9881	2.66	.9961	3.06	.9989	3.46	.9997
1.87	.9693	2.27	.9884	2.67	.9962	3.07	.9989	3.47	.9997
1.88	.9699	2.28	.9887	2.68	.9963	3.08	.9990	3.48	.9997
1.89	.9706	2.29	.9890	2.69	.9964	3.09	.9990	3.49	.9998
1.90	.9713	2.30	.9893	2.70	.9965	3.10	.9990	3.50	.9998
1.91	.9719	2.31	.9896	2.71	.9966	3.11	.9991	3.51	.9998
1.92	.9726	2.32	.9898	2.72	.9967	3.12	.9991	3.52	.9998
1.93	.9732	2.33	.9901	2.73	.9968	3.13	.9991	3.53	.9998
1.94	.9738	2.34	.9904	2.74	.9969	3.14	.9992	3.54	.9998
1.95	.9744	2.35	.9906	2.75	.9970	3.15	.9992	3.55	.9998
1.96	.9750	2.36	.9909	2.76	.9971	3.16	.9992	3.56	.9998
1.97	.9756	2.37	.9911	2.77	.9972	3.17	.9992	3.57	.9998
1.98	.9761	2.38	.9913	2.78	.9973	3.18	.9993	3.58	.9998
1.99	.9767	2.39	.9916	2.79	.9974	3.19	.9993	3.59	.9998
2.00	.9772	2.40	.9918	2.80	.9974	3.20	.9993	3.60	.9998
2.01	.9778	2.41	.9920	2.81	.9975	3.21	.9993	3.61	.9998
2.02	.9783	2.42	.9922	2.82	.9976	3.22	.9994	3.62	.9999
2.03	.9788	2.43	.9925	2.83	.9977	3.23	.9994	3.63	.9999
2.04	.9793	2.44	.9927	2.84	.9977	3.24	.9994	3.64	.9999
2.05	.9798	2.45	.9929	2.85	.9978	3.25	.9994	3.65	.9999
2.06	.9803	2.46	.9931	2.86	.9979	3.26	.9994	3.66	.9999
2.07	.9808	2.47	.9932	2.87	.9979	3.27	.9995	3.67	.9999
2.08	.9812	2.48	.9934	2.88	.9980	3.28	.9995	3.68	.9999
2.09	.9817	2.49	.9936	2.89	.9981	3.29	.9995	3.69	.9999
2.10	.9821	2.50	.9938	2.90	.9981	3.30	.9995	3.70	.9999
2.11	.9826	2.51	.9940	2.91	.9982	3.31	.9995	3.71	.9999
2.12	.9830	2.52	.9941	2.92	.9982	3.32	.9996	3.72	.9999
2.13	.9834	2.53	.9943	2.93	.9983	3.33	.9996	3.73	.9999
2.14	.9838	2.54	.9945	2.94	.9984	3.34	.9996	3.74	.9999
2.15	.9842	2.55	.9946	2.95	.9984	3.35	.9996	3.75	.9999
2.16	.9846	2.56	.9948	2.96	.9985	3.36	.9996	3.76	.9999
2.17	.9850	2.57	.9949	2.97	.9985	3.37	.9996	3.77	.9999
2.18	.9854	2.58	.9951	2.98	.9986	3.38	.9996	3.78	.9999
2.19	.9857	2.59	.9952	2.99	.9986	3.39	.9997	3.79	.9999
2.20	.9861	2.60	.9953	3.00	.9986	3.40	.9997	3.80	.9999

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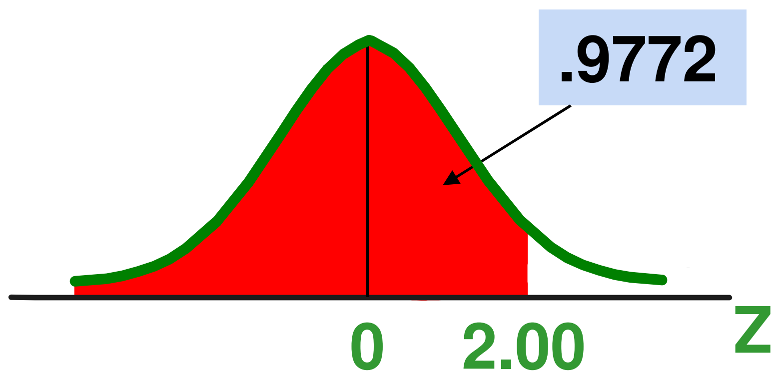
The Standardized Normal Table

- Appendix Table 1 gives the probability Φ(a) for any value a

Example:

P(Z < 2.00) = .9772



The Standardized Normal Table

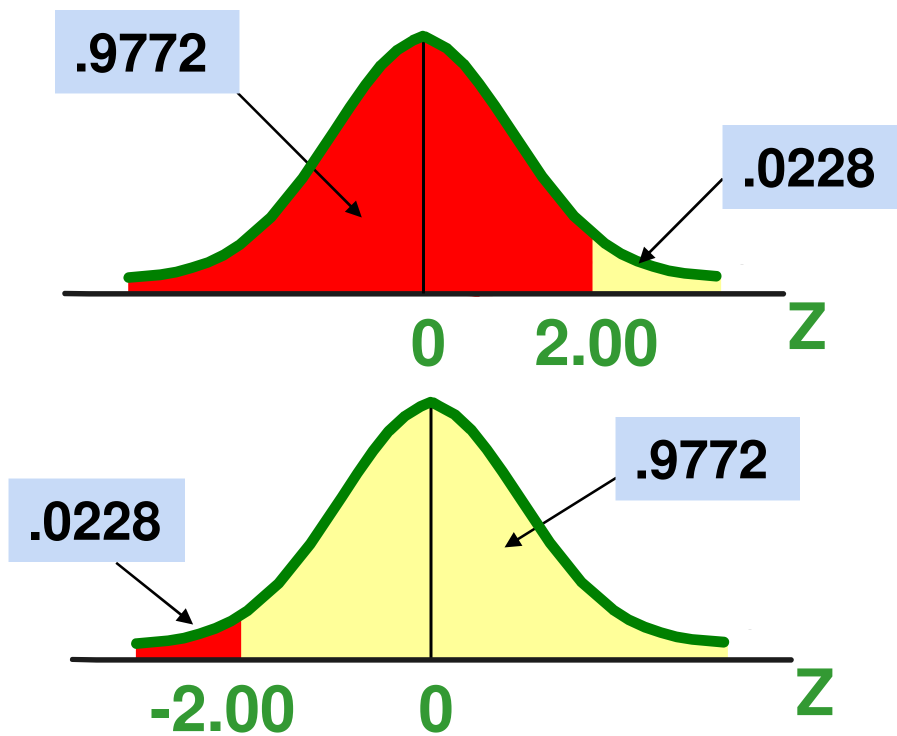
(continued)

- For negative Z-values, use the fact that the distribution is symmetric to find the needed probability:

Example:

P(Z < -2.00) = 1 – 0.9772

= 0.0228



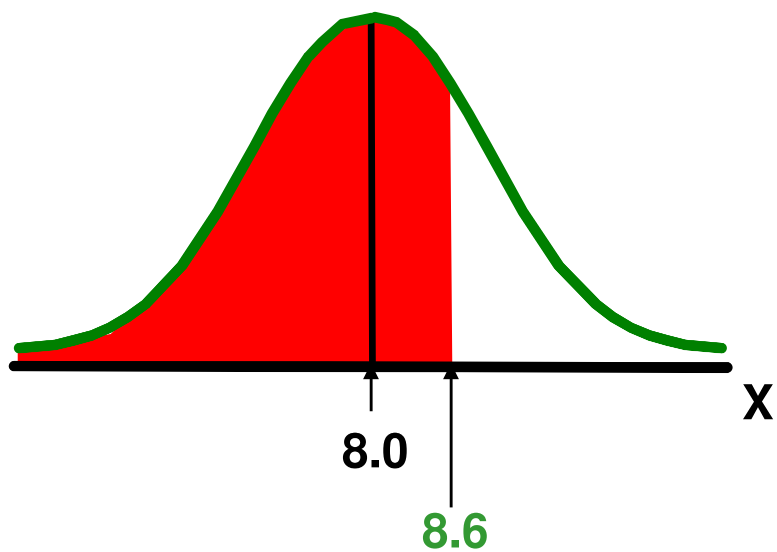
General Procedure for Finding Probabilities

To find $P(a < X < b)$ when X is distributed normally:

- Draw the normal curve for the problem in terms of X
- Translate X-values to Z-values
- Use the Cumulative Normal Table

Finding Normal Probabilities

- Suppose X is normal with mean 8.0 and standard deviation 5.0
- Find P(X < 8.6)

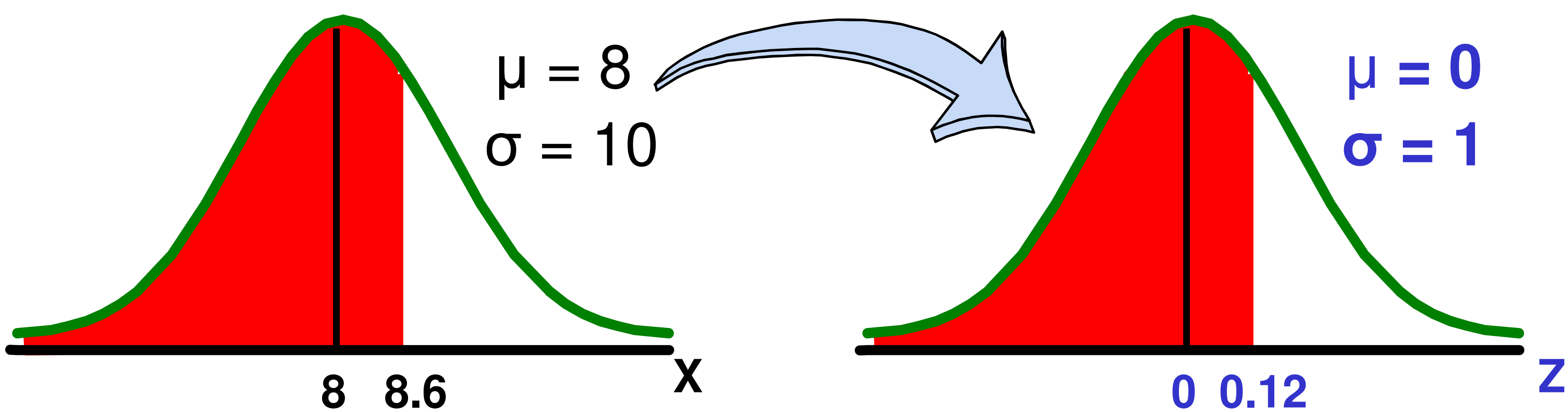


Finding Normal Probabilities

(continued)

- Suppose X is normal with mean 8.0 and standard deviation 5.0. Find P(X < 8.6)

$$Z = \frac{X - \mu}{\sigma} = \frac{8.6 - 8.0}{5.0} = 0.12$$



P(X < 8.6)

P(Z < 0.12)

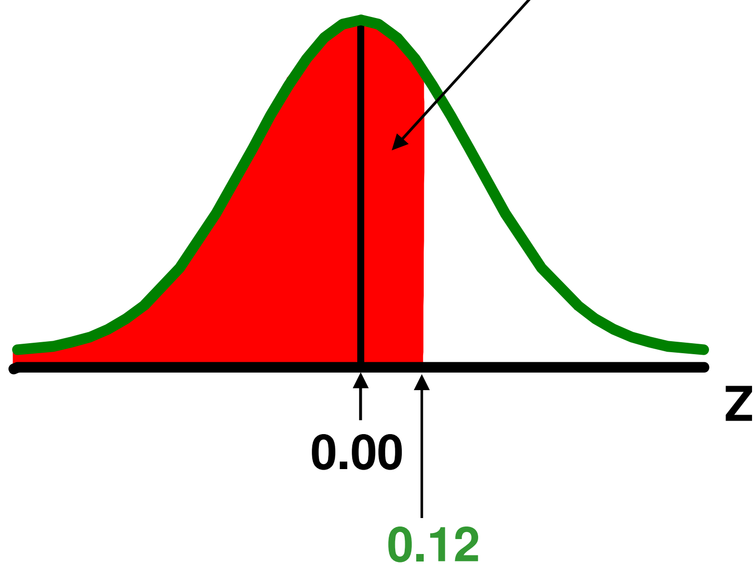
Solution: Finding P(Z < 0.12)

Standardized Normal Probability Table (Portion)

z	Φ(z)
.10	.5398
.11	.5438
.12	.5478
.13	.5517

$P(X < 8.6)$
$= P(Z < 0.12)$

$\Phi(0.12) = 0.5478$



Clicker Question 5-4

- Suppose X is distributed normally with mean of 20 and standard deviation of 5, what is $P(X<29.8)$?

A). 0.9

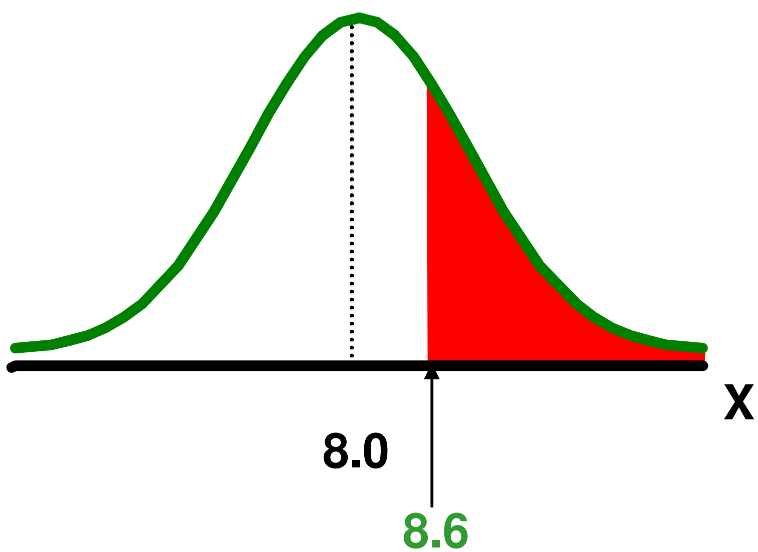
B). 0.95

C). 0.975

D). 0.99

Upper Tail Probabilities

- Suppose X is normal with mean 8.0 and standard deviation 5.0.
- Now Find P(X > 8.6)

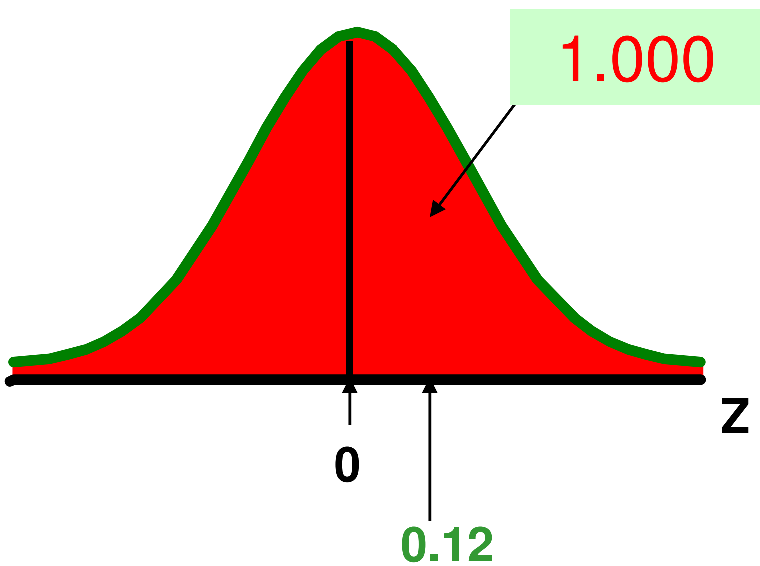


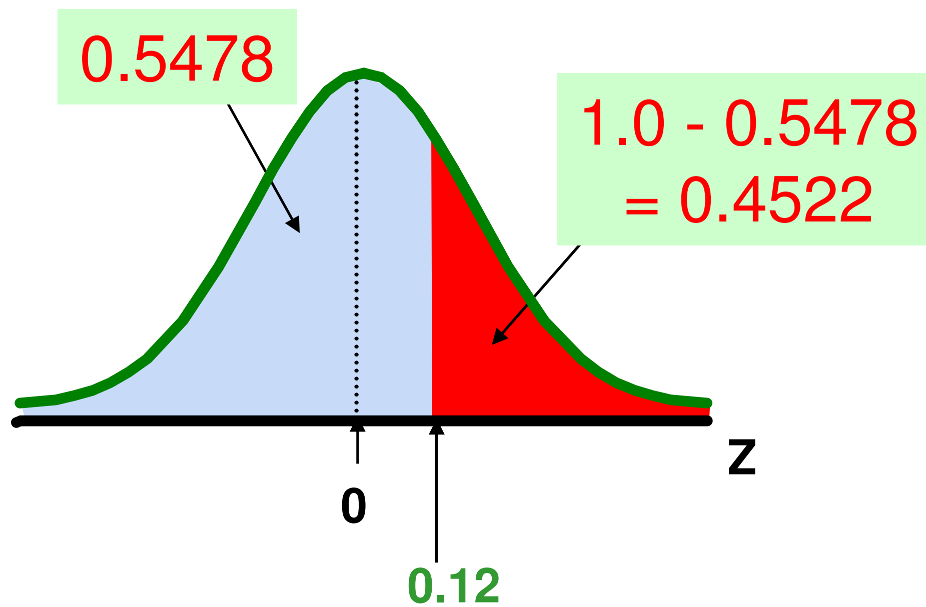
Upper Tail Probabilities

(continued)

- Now Find P(X > 8.6)…

$$P(X > 8.6) = P(Z > 0.12) = 1.0 - P(Z \le 0.12)$$
$$= 1.0 - 0.5478 = 0.4522$$





Clicker Question 5-5

- Suppose X is distributed normally with mean of 20 and standard deviation of 5, what is P(X>29.8)?

A). 0.1

B). 0.05

C). 0.025

D). 0.01

Finding the X value for a Known Probability

- Steps to find the X value for a known probability:
  1. Find the Z value for the known probability
  2. Convert to X units using the formula:

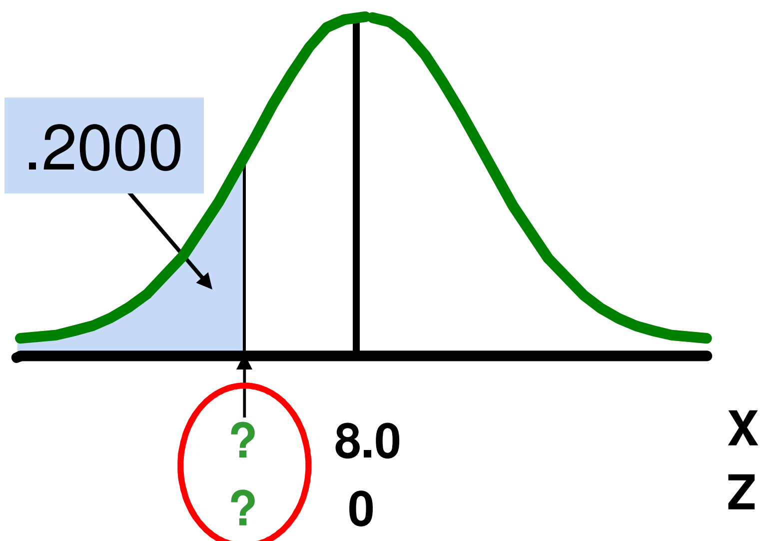
$$X = \mu + Z\sigma$$

Finding the X value for a Known Probability

(continued)

Example:

- Suppose X is normal with mean 8.0 and standard deviation 5.0.
- Now find the X value so that only 20% of all values are below this X



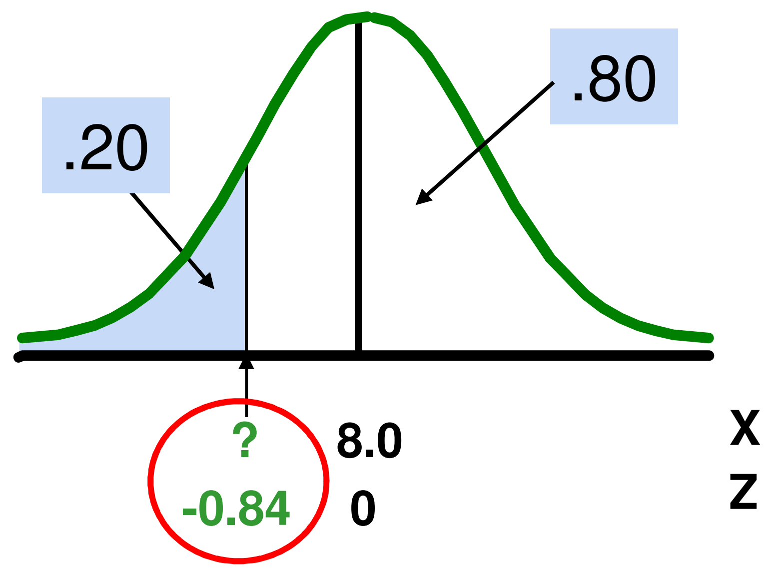
Find the Z value for 20% in the Lower Tail

1. Find the Z value for the known probability

Standardized Normal Probability Table (Portion)

z	Φ(z)
.82	.7939
.83	.7967
.84	.7995
.85	.8023

- 20% area in the lower tail is consistent with a Z value of -0.84



Finding the X value

2. Convert to X units using the formula:

$$X = \mu + Z\sigma$$

$$= 8.0 + (-0.84)5.0$$

$$= 3.80$$

So 20% of the values from a distribution with mean 8.0 and standard deviation 5.0 are less than 3.80

Clicker Question 5-5

- Suppose X is distributed normally with mean of 20 and standard deviation of 5, what is the value of x such that $P(X<x)=0.025$?

A). 10.20

B). 11.78

C). 13.60

5.6

Joint Cumulative Distribution Functions

- Let $X_1, X_2, \ldots X_k$ be continuous random variables

- Their joint cumulative distribution function,

  $F(x_1, x_2, \ldots x_k)$

  defines the probability that simultaneously X_1 is less than x_1, X_2 is less than x_2, and so on; that is

$$F(x_1, x_2, \ldots, x_k) = P(\{X_1 < x_1\} \cap \{X_2 < x_2\} \cap \cdots \{X_k < x_k\})$$

Joint Cumulative Distribution Functions

(continued)

- The cumulative distribution functions

  $$F(x_1), F(x_2), . . ., F(x_k)$$

  of the individual random variables are called their marginal distribution functions

- The random variables are independent if and only if

$$F(x_1, x_2, \ldots, x_k) = F(x_1)F(x_2)\cdots F(x_k)$$

Covariance

- Let X and Y be continuous random variables, with means μ_x and μ_y
- The expected value of $(X - \mu_x)(Y - \mu_y)$ is called the covariance between X and Y

$$\mathrm{Cov}(X,Y) = E[(X - \mu_x)(Y - \mu_y)]$$

- An alternative but equivalent expression is

$$\mathrm{Cov}(X,Y) = E(XY) - \mu_x\mu_y$$

- If the random variables X and Y are independent, then the covariance between them is 0. However, the converse is not true.

Correlation

- Let X and Y be jointly distributed random variables.
- The correlation between X and Y is

$$\rho = \mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

Sums of Random Variables

Let $X_1, X_2, \ldots X_n$ be n random variables with means $\mu_1, \mu_2, \ldots \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$. Then:

- The mean of their sum is the sum of their means

$$E(X_1 + X_2 + \cdots + X_n) = \mu_1 + \mu_2 + \cdots + \mu_n$$

Sums of Random Variables

(continued)

Let $X_1, X_2, \ldots X_n$ be n random variables with means $\mu_1, \mu_2, \ldots \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$.

$$\mathrm{Var}(X_1 + X_2 + \cdots + X_n) = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\mathrm{Cov}(X_i, X_j)$$

If the covariance between every pair of these random variables is 0,

$$\mathrm{Var}(X_1 + X_2 + \cdots + X_n) = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2$$

ii.

Clicker Question 5-6

Let $X_1, X_2, \ldots X_n$ be n random variables that are independent with identical mean and variance, i.e., $\mathrm{E}[X_i] = \mu$ and $Var[X_i] = \sigma^2$ for $i = 1,2 \ldots, n$. What is the variance of $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$?

A). $\mathrm{Var}(\bar{X}) = \sigma^2$

B). $\mathrm{Var}(\bar{X}) = \sigma^2/n$

C). $\mathrm{Var}(\bar{X}) = \sigma^2/n^2$

Linear Combinations of Random Variables

- A linear combination of two random variables, X and Y, (where a and b are constants) is

$$W = aX + bY$$

- The mean of W is

$$\mu_W = E[W] = E[aX + bY] = a\mu_X + b\mu_Y$$

Linear Combinations of Random Variables

(continued)

- The variance of W is

$$\sigma_W^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\mathrm{Cov}(X,Y)$$

- Or using the correlation,

$$\sigma_W^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\mathrm{Corr}(X,Y)\sigma_X\sigma_Y$$

Linear combination of normal random variables

- When X and Y are jointly normally distributed, the linear combination of X and Y is also jointly normally distributed, i.e.,

$$aX + bY \sim N(a\mu_X + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2 + 2abCov(X, Y))$$

Linear combination of normal random variables

- Let $X_1, X_2, \ldots X_n$ be n **normally distributed** random variables that are independent with identical mean and variance, i.e., $\mathrm{E}[X_i] = \mu$ and $Var[X_i] = \sigma^2$ for $i = 1,2 \ldots, n$.
- $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$
- Then,

$$\overline{X} \sim N\left(\mu, \sigma^2/n\right)$$

Portfolio Analysis

- A financial portfolio can be viewed as a linear combination of separate financial instruments

$$\begin{pmatrix}\text{Return on}\\ \text{portfolio}\end{pmatrix} = \begin{pmatrix}\text{Proportion of}\\ \text{portfolio value}\\ \text{in stock1}\end{pmatrix} \times \begin{pmatrix}\text{Stock 1}\\ \text{return}\end{pmatrix} + \begin{pmatrix}\text{Proportion of}\\ \text{portfolio value}\\ \text{in stock2}\end{pmatrix} \times \begin{pmatrix}\text{Stock 2}\\ \text{return}\end{pmatrix}$$

$$\cdots + \begin{pmatrix}\text{Proportion of}\\ \text{portfolio value}\\ \text{in stock N}\end{pmatrix} \times \begin{pmatrix}\text{Stock N}\\ \text{return}\end{pmatrix}$$

Portfolio Analysis Example

- Consider two stocks, A and B
  - The price of Stock A is normally distributed with mean 12 and standard deviation 4
  - The price of Stock B is normally distributed with mean 20 and standard deviation 16
  - The stock prices have a positive correlation, $\rho_{AB} = .50$
- Suppose you own
  - 10 shares of Stock A
  - 30 shares of Stock B

Portfolio Analysis Example

(continued)

- The mean and variance of this stock portfolio are: (Let W denote the distribution of portfolio value)

$$\mu_W = 10\mu_A + 20\mu_B = (10)(12) + (30)(20) = 720$$

$$\begin{aligned}\sigma_W^2 &= 10^2\sigma_A^2 + 30^2\sigma_B^2 + (2)(10)(30)\text{Corr}(A,B)\sigma_A\sigma_B \\ &= 10^2\,(4)^2 + 30^2(16)^2 + (2)(10)(30)(.50)(4)(16) \\ &= 251{,}200\end{aligned}$$

Portfolio Analysis Example

(continued)

- What is the probability that your portfolio value is less than $500?

$$\mu_W = 720$$

$$\sigma_W = \sqrt{251{,}200} = 501.20$$

- The Z value for 500 is $Z = \dfrac{500 - 720}{501.20} = -0.44$

- P(Z < -0.44) = 0.3300
  - So the probability is 0.33 that your portfolio value is less than $500.