Econ 325 Section 001
Assignment 4

The due date is Tuesday, October 15, before the class starts.

1. Let X and Y be two discrete random variables. The set of possible values for X is $\{x_1, \ldots, x_n\}$; and the set of possible values for Y is $\{y_1, \ldots, y_m\}$. The joint pmf (probability mass function) is given by

$$p_{ij}^{X,Y} = P\left(X = x_i, Y = y_j\right), \qquad i = 1, \ldots n; j = 1, \ldots, m.$$

The marginal pmf of X is

$$p_i^X = P\left(X = x_i\right) = \sum_{j=1}^{m} p_{ij}^{X,Y}, \qquad i = 1, \ldots n,$$

and the marginal pmf of Y is

$$p_j^Y = P\left(Y = y_j\right) = \sum_{i=1}^{n} p_{ij}^{X,Y}, \qquad i = 1, \ldots n,$$

By definition of conditional mass function, we can express the conditional mass function of Y given $X = x$ as $P\left(Y = y_j | X = x_i\right) = \frac{p_{ij}^{X,Y}}{p_i^X}$. Please use summation operator for proof whenever possible. Let a, b, and c be constant.

(a) Prove that, if X and Y are stochastically independent, then $Cov(g(X), Y) = 0$ for any function g.

(b) Let $g_1(x)$ and $g_2(x)$ be some functions of x. Prove that $Var\left(g_1(X) + g_2(X)\right) = Var\left(g_1(X)\right) + Var\left(g_2(X)\right) + 2Cov\left(g_1(X), g_2(X)\right)$.

(c) Let b be a constant. Show that $E[(X - b)^2] = E(X^2) - 2bE(X) + b^2$. What is the constant value of b that gives the minimum value of $E[(X - b)^2]$?

(d) Define $Z = (X - E(X))/\sqrt{Var(X)}$. Prove that $E[Z] = 0$ and $Var[Z] = 1$.

(e) Define $Z = (X - E(X))/\sqrt{Var(X)}$. Prove that $\text{Corr}(X, Z) = 1$.

(f) Consider another random variable Z in addition to X and Y. Prove that $Var(aX + bY + cZ) = a^2Var(X) + b^2Var(Y) + c^2Var(Z) + 2abCov(X, Y) + 2acCov(X, Z) + 2bcCov(Y, Z)$ for any constant a, b, and c.

(g) Show that $Corr(X, Y) = -1$ or 1 if $Y = a + bX$.

(h) Show that $E_X[E_Y[Y|X]] = E_Y[Y]$.

2. Suppose that $\{X_1, X_2, ..., X_n\}$ is a random sample, where X_i takes a value of zero or one with probability $1 - p$ and p, respectively. Define $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Show that

$$E(\bar{X}) = p \quad \text{and} \quad \text{Var}(\bar{X}) = \frac{p(1-p)}{n}.$$

3. The number of hits per day on the Web site of Professional Tool, Inc., is normally distributed with a mean of 700 and a standard deviation of 120.

   (a) What proportion of days has more than 820 hits per day?

   (b) What proportion of days has between 730 and 820 hits?

   (c) Find the number of hits such that only 5% of the days will have the number of hits below this number.