Econ 325 Section 001
Assignment 5

The due date is Tuesday, October 29 before the class starts.

1. Given a population with a mean of $\mu = 100$ and a variance of $\sigma^2 = 81$, suppose that the central limit theorem applies when the sample size is $n = 25$. A random sample of size n = 25 is obtained. Let $\bar{X}$ be the sample mean of $n = 25$ observations.
   (a) What are the mean and variance of the sampling distribution for the sample means?
   (b) What is the probability that $\bar{X} > 102$?
   (c) What is the probability that $98 \leq \bar{X} \leq 101$?
   (d) Find the value of a such that $P(100 - a \leq \bar{X} \leq 100 + a) = 090$.
   (e) Find the value of b such that $P(100 - b \leq \bar{X} \leq 100 + b) = 095$.
   (f) Find the value of c such that $P(100 - c \leq \bar{X} \leq 100 + c) = 099$.

2. A corporation receives 120 applications for positions from recent college graduates in business. We assume that these applicants can be viewed as a random sample of all such graduates. 40% of all recent college graduates in business are women . Answer the following questions with a normal approximation by the central limit theorem.
   (a) What is the probability that between 35% and 45% of the 120 applications are women?
   (b) Let $\hat{p}$ be the fraction of women in the 120 applications and let p be the fraction of women in all recent college graduates in business. What is the value of a such that $P(p - a \leq \hat{p} \leq p + a) = 0.95$?

3. Suppose that $\{X_1, X_2, ..., X_n\}$ is a random sample from a population, where $E[X_i] = \mu$ and $\text{Var}[X_i] = \sigma^2$. We do not assume normality. Define the sample mean of X_i's as $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.
   (a) Show that
   $$E(\bar{X}) = \mu \quad \text{and} \quad \sqrt{Var(\bar{X})} = \frac{\sigma}{\sqrt{n}}.$$
   (b) What is the variance of $\sqrt{n}(\bar{X} - \mu)$? What is the limit of $Var(\sqrt{n}(\bar{X} - \mu))$ as $n \to \infty$.
   (c) What is the variance of $n^{1/4}(\bar{X} - \mu)$? What is the limit of $Var(n^{1/4}(\bar{X} - \mu))$ as $n \to \infty$.
   (d) What is the variance of $n(\bar{X} - \mu)$? What is the limit of $Var(n(\bar{X} - \mu))$ as $n \to \infty$.

4. Suppose that $\{X_1, X_2, ..., X_{n_x}\}$ is a random sample with sample size n_x, where X_i takes a value of zero or one with probability $1 - p_x$ and p_x, respectively, and that $\{Y_1, Y_2, ..., Y_{n_y}\}$ is a random sample of sample size n_y, where Y_i takes a value of zero or one with probability $1 - p_y$ and p_y, respectively. Note that X_i and Y_j for any i and j are stochastically independent by random sampling. Derive $E(\bar{X} - \bar{Y})$ and $\text{Var}(\bar{X} - \bar{Y})$.

5. If Z is $N(0,1)$, then $P(|Z| < 1.96) = 0.95$. Using the fact that Z^2 is the chi-square distributed with the degree of freedom 1, i.e., $Z^2 = \chi_1^2$, derive the value of a in the following equation without looking at the chi-square table:

$$P(\chi_1^2 < a) = 0.95.$$

Confirm that your guess on the value of a is correct by checking the chi-square table with $\nu = 1$. Can you also guess what would be the value of b for $P(\chi_1^2 < b) = 0.90$ by using the fact that $P(|Z| < 1.645) = 0.90$?

6. A company produces electrical devices operated by a thermostatic control. According to the engineering specifications, the variance of the temperature at which these controls actually operate should not exceed 4.0 degrees Fahrenheit. Suppose that we have a random sample of 25 of these controls.

   (a) Determine the upper limit for the sample variance such that the probability of exceeding this limit is less than 0.05 when the temperature is normally distributed with the population variance equal to 4.

   (b) Suppose that the sample variance of 25 observations is equal to 10. Do you think that this provides evidence for or against the population variance to be smaller than 4?