Econ 325
Notes on Bernoulli Random Variable and Binomial Distribution[1]
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Bernoulli Random Variable

Consider a random variable X that takes a value of zero or one with probability $1-p$ and p, respectively. That is,

$$X = \begin{cases} 0 & \text{with prob. } 1-p \\ 1 & \text{with prob. } p \end{cases} \tag{1}$$

The probability mass function is written as $f(x) = p^x(1-p)^{1-x}$ and we say that X has a Bernoulli distribution.
The expected value of X is

$$E(X) = \sum_{x=0,1} xp^x(1-p)^{1-x} = (0)(1-p) + (1)(p) = p$$

and the variance of X is

$$\begin{aligned}\text{Var}(X) &= \sum_{x=0,1} (x-p)^2 p^x(1-p)^{1-x} \\ &= (0-p)^2(1-p) + (1-p)^2 p \\ &= p^2(1-p) + (1-p)^2 p = (p + (1-p)) \times p(1-p) = p(1-p).\end{aligned}$$

The standard deviation of X is $\sqrt{\text{Var}(X)} = \sqrt{p(1-p)}$.

Binomial Distribution

Let X_1, X_2,..., X_n be a sequence of n independent Bernoulli random variables, each of which has the probability of success equal to p, given by (1). Define a random variable Y by

$$Y = \sum_{i=1}^{n} X_i,$$

i.e., Y is the number of successes in n Bernoulli trials. The number of ways of selecting y positions for the y successes in the n trials is

$$\binom{n}{y} = \frac{n!}{y!(n-y)!}.$$

Then, the probability mass function of Y is given by

$$f(y) = \binom{n}{y} p^y (1-p)^{(n-y)}. \tag{2}$$

The expected value of $Y = \sum_{i=1}^n X_i$ is

$$E(Y) = E(\sum_{i=1}^n X_i) = \sum_{i=1}^n E(X_i) = np$$

where the last equality follows because $E(X_i) = p$ which is a constant and $\sum_{i=1}^n c = nc$ for any constant c.
Let $Z_i = X_i - p$. Then, the variance of $Y = \sum_{i=1}^n X_i$ is

$$\begin{aligned}
\text{Var}(Y) &= E[\{(\sum_{i=1}^n X_i) - np\}^2] \\
&= E[\{\sum_{i=1}^n (X_i - p)\}^2] \\
&= E[\{\sum_{i=1}^n Z_i\}^2] \qquad (\text{Define } Z_i = X_i - p \text{ for } i = 1, ..., n) \\
&= E[Z_1^2 + Z_2^2 + ... + Z_n^2 + 2Z_1Z_2 + 2Z_1Z_3 + ... + 2Z_1Z_n + ... + 2Z_{n-1}Z_n] \\
&= \sum_{i=1}^n E[Z_i^2] + 2\sum_{i=1}^n \sum_{j=i+1}^n E[Z_iZ_j] \\
&= \sum_{i=1}^n p(1-p) + 2 \times 0 \qquad (\text{Because } E[Z_i^2] = \text{Var}(X_i) = p(1-p) \text{ and } E[Z_iZ_j] = 0 \text{ if } i \neq j) \\
&= np(1-p),
\end{aligned}$$

where $E[Z_iZ_j] = 0$ if $i \neq j$ because X_i and X_j is independent if $i \neq j$.

The Sample Mean of Bernoulli Random Variables

Binomial distribution is closely related to the distribution of the sample mean of Bernoulli random variables. Define

$$\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i,$$

where X_1, X_2,..., X_n are a sequence of n independent Bernoulli random variables. Then, the possible values $\bar{X}$ can take are $\{0, 1/n, 2/n, ..., (n-1)/n, 1\}$. Further, $\bar{X} = (1/n)Y$ so that $Y = n\bar{X}$. Therefore, by letting $y = n\bar{x}$ in (2), the probability mass function of $\bar{X}$ is given by

$$\Pr(\bar{X} = \bar{x}) = \binom{n}{(n\bar{x})} p^{n\bar{x}}(1-p)^{(n-n\bar{x})}.$$

This is the exact probability mass function of $\bar{X}$ when X_is are independent Bernoulli random variables. Later, we will discuss that the distribution of $\bar{X}$ can be *approximated* by the normal distribution when n is large. We may also compute the expected value and the variance of $\bar{X}$ from those of $Y = \sum_{i=1}^n X_i$. In fact,

$$E[\bar{X}] = E[(1/n)Y] = (1/n)E[Y] = (1/n)np = p,$$

and

$$Var(\bar{X}) = Var((1/n)Y) = (1/n)^2 Var(Y) = (1/n)^2 np(1-p) = \frac{p(1-p)}{n}.$$

Examples

1. Flip a coin four times and let Y be the number of heads. What is the probability that the number of heads Y is equal to 2?

   Answer: We have $n = 4$ and $p = 0.5$. The probability that $Y = 2$ is equal to

$$\binom{4}{2}(0.5)^2(1-0.5)^{(4-2)} = 6(0.5)^4 = \frac{3}{8}.$$

2. Exercise 4.40: The Minnesota Twins are to play a series of 5 games against the Red Sox. For any one game it is estimated that the probability of a Twins' win is 0.5. The outcome of the 5 games are independent of one another. (a) What is the probability that Twins will win all 5 games? (b) What is the probability that Twins will win a majority of the 5 games? (c) If the Twins win the first game, what is the probability that they will win a majority of the five games?

   Answer: We have $n = 5$ and $p = 0.5$. For (a), the probability that $Y = 5$ is equal to

$$\Pr(Y = 5) = \binom{5}{5}(0.5)^5(1-0.5)^{(5-5)} = 1 \times (0.5)^5 \times 1 = \frac{1}{32}.$$

   For (b), the probability that $Y = 3$ and that of $Y = 4$ are

$$\Pr(Y = 3) = \binom{5}{3}(0.5)^3(1-0.5)^{(5-3)} = 10 \times (0.5)^5 = \frac{10}{32},$$
$$\Pr(Y = 4) = \binom{5}{4}(0.5)^4(1-0.5)^{(5-4)} = 5 \times (0.5)^5 = \frac{5}{32}.$$

   Therefore, the probability that Twins will win a majority of the 5 games, namely, $Y \geq 3$ is

$$\Pr(Y \geq 3) = \Pr(Y = 3) + \Pr(Y = 4) + \Pr(Y = 5) = \frac{10}{32} + \frac{5}{32} + \frac{1}{32} = \frac{1}{2}.$$

   For (c), when the Twins win the first game, the Twins need to win at least 2 games out of four games to win a majority. Because the outcomes of the 5 games are independent of one another, we may define a new Binomial random variable $W = \sum_{i=1}^{n} X_i$ with $\underline{n = 4}$ and $p = 0.5$, which represents the outcomes of the 4 games after the first game.

$$\Pr(W \geq 2) = \Pr(W = 2) + \Pr(W = 3) + \Pr(W = 4) = \frac{6}{16} + \frac{4}{16} + \frac{1}{16} = \frac{11}{16},$$

   where, for example, $\Pr(W = 2) = \binom{4}{2}(0.5)^2(1-0.5)^{(4-2)} = 6(0.5)^4 = 6/16$. Therefore, the probability for the Twins to win a majority after winning the first game is $11/16$.

3. Let X_1 and X_2 are two Bernoulli random variables with the probability of success p, where X_1 and X_2 are independent, and $X_i = 0$ with probability $1 - p$ and $X_i = 1$ with probability p for $i = 1, 2$. Define a random variable $Y = X_1 + X_2$. Therefore, Y follows the Binomial Distribution with $n = 2$ trials.

(a) Find the mean and the variance of Y.

Answer: Note that $E[X_1] = E[X_2] = p$ and $\text{Var}(X_1) = \text{Var}(X_2) = p(1-p)$ because $E[X_i] = 0 \times (1-p) + 1 \times p = p$ and $Var(X_i) = E[X_i^2] - \{E[X_i]\}^2 = p - p^2 = p(1-p)$, where $E[X_i^2] = p$ follows from $E[X_i^2] = E[X_i] = p$ with $X_i^2 = X_i$. Furthermore, because X_1 and X_2 are independent, $\text{Cov}(X_1, X_2) = 0$. Therefore, $E[Y] = E[X_1 + X_2] = E[X_1] + E[X_2] = p + p = 2p$ and $\text{Var}(Y) = \text{Var}(X_1 - X_2) = \text{Var}(X_1) + \text{Var}(X_2) + 2\text{Cov}(X_1, X_2) = p(1-p) + p(1-p) + 0 = 2p(1-p)$.

(b) What is $E[Y|X_1 = 1]$?

Answer: $E[Y|X_1 = 1] = E[X_1 + X_2|X_1 = 1] = E[X_1|X_1 = 1] + E[X_2|X_1 = 1] = 1 + E[X_2] = 1 + p$, where $E[X_1|X_1 = 1] = 1$ holds $X_1 = 1$ with probability one when we condition on the event that $X_1 = 1$ and $E[X_2|X_1 = 1] = E[X_2]$ because X_1 and X_2 are independent.

(c) What is $E[X_1|Y = 1]$?

Answer: $\Pr(X_1 = 1|Y = 1) = \Pr(X_1 = 1|X_1 + X_2 = 1) = \frac{\Pr(X_1=1, X_1+X_2=1)}{\Pr(X_1+X_2=1)}$. Note that there are four possible outcomes for (X_1, X_2): $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$. Now, $\Pr(X_1 = 1, X_1 + X_2 = 1) = \Pr(X_1 = 1, X_2 = 0) = p(1-p)$ and $\Pr(X_1 + X_2 = 1) = \Pr(X_1 = 0, X_2 = 1) + \Pr(X_1 = 1, X_2 = 0) = 2p(1-p)$. Therefore, $\Pr(X_1 = 1|Y = 1) = \frac{\Pr(X_1=1, X_1+X_2=1)}{\Pr(X_1+X_2=1)} = \frac{p(1-p)}{2p(1-p)} = 1/2$. Similarly, we may prove that $\Pr(X_1 = 0|Y = 1)$. Finally, $E[X_1|Y = 1] = E[X_1|X_1 + X_2 = 1] = \sum_{x_1 \in \{0,1\}} x_1 \Pr(X_1 = x_1|X_1 + X_2 = 1) = 0 \times (1/2) + 1 \times (1/2) = 1/2$.