Econ 326 Section 004

Notes on Summation Operator

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Summation Operator

The summation operator, represented by the upper-case Greek letter sigma Σ, is a shorthand notation to represent the sum of a sequence of numbers, such as x_1, x_2,... x_n. For a sequence of the values $\{x_1, x_2, ..., x_n\}$, we write the sum of x_1, x_2, ..., x_{n-1}, and x_n using the summation operator as

$$x_1 + x_2 + ... + x_n = \sum_{i=1}^{n} x_i. \tag{1}$$

For example, consider $x_1 = 2$, $x_2 = 6$, $x_3 = 1$, $x_4 = 12$, and $x_5 = 3$ and we can write the sum of x_1, x_2,... x_5 using the summation operator as

$$\sum_{i=1}^{5} x_i = 2 + 6 + 1 + 12 + 3 = 24.$$

Given a constant c,

$$\sum_{i=1}^{n} cx_i = cx_1 + cx_2 + ... + cx_n = c \times (x_1 + x_2 + ... + x_n) = c \sum_{i=1}^{n} x_i. \tag{2}$$

- For example, consider the case that $n = 2$ with the values of $\{x_1, x_2\}$ given by $x_1 = 0$ and $x_2 = 1$. Suppose that $c = 4$. Then, $\sum_{i=1}^{2} 4 \times x_i = 4 \times 0 + 4 \times 1 = 4 \times (0 + 1) = 4 \sum_{i=1}^{2} x_i$.
- In the special case of $x_1 = x_2 = ... = x_n = 1$, we have

$$\sum_{i=1}^{n} c = \sum_{i=1}^{n} c \times 1 = c \times \sum_{i=1}^{n} 1 = c \times (1 + 1 + ... + 1) = nc.$$

Consider another sequence $\{y_1, y_2, ..., y_m\}$ in addition to $\{x_1, x_2, ..., x_n\}$. Then, we may consider double summations over possible values of x's and y's. For example, consider the case of $n = m = 2$. Then,

$$\sum_{i=1}^{2} \sum_{j=1}^{2} x_i y_j = x_1 y_1 + x_1 y_2 + x_2 y_1 + x_2 y_2$$

because

$$
\begin{aligned}
& x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2 \\
& = x_1(y_1+y_2) + x_2(y_1+y_2) \quad \text{(by factorization)} \\
& = \sum_{i=1}^{2} x_i(y_1+y_2) \quad \text{(by def. of the summation operator by setting } c=(y_1+y_2) \text{ in (2))} \\
& = \sum_{i=1}^{2} x_i \left(\sum_{j=1}^{2} y_j\right) \quad \text{(because } y_1+y_2=\textstyle\sum_{j=1}^{2} y_j\text{)} \\
& = \sum_{i=1}^{2} \left(\sum_{j=1}^{2} x_iy_j\right) \quad \text{(because } x_i\textstyle\sum_{j=1}^{2} y_j = x_i(y_1+y_2) = (x_iy_1+x_iy_2) = \textstyle\sum_{j=1}^{2} x_iy_j\text{)} \\
& = \sum_{i=1}^{2}\sum_{j=1}^{2} x_iy_j.
\end{aligned}
$$

- Note that $\sum_{i=1}^{2}\sum_{j=1}^{2} x_iy_j = \sum_{j=1}^{2}\sum_{i=1}^{2} x_iy_j$. In general case of $\{x_1, x_2, ..., x_n\}$ and $\{y_1, y_2, ..., y_m\}$, we have $\sum_{i=1}^{n}\sum_{j=1}^{m} x_iy_j = \sum_{j=1}^{m}\sum_{i=1}^{n} x_iy_j$.
- Note that $\sum_{j=1}^{2} x_iy_j = x_i\sum_{j=1}^{2} y_j$ using (2) because x_i is treated as a constant in the summation operator over j's. Hence, we can write

  $$\sum_{i=1}^{2}\sum_{j=1}^{2} x_iy_j = \sum_{i=1}^{2} x_i \sum_{j=1}^{2} y_j = \sum_{j=1}^{2} y_j \sum_{i=1}^{2} x_i.$$

  In general, we have

  $$\sum_{i=1}^{n}\sum_{j=1}^{m} x_iy_j = \sum_{i=1}^{n} x_i \sum_{j=1}^{m} y_j = \sum_{j=1}^{m} y_j \sum_{i=1}^{n} x_i. \tag{3}$$

  That is, when we have double summations, we can take x_i's out of the summation over j's. Similarly, we can take y_j's out of the summation over i's.

Examples

- The final grade point in Econ 325 is computed as a weighted average of assignment (x_1), midterm (x_2), and final exam (x_3). The weights are $w_1 = 0.1$, $w_2 = 0.3$, and $w_3 = 0.6$ for assignment, midterm, and final exam, respectively. Using the summation sign, we may express the final grade point in terms of x_i and w_i for $i = 1, 2, 3$ as

  $$\sum_{i=1}^{n} w_ix_i = w_1x_1 + w_2x_2 + w_3x_3$$

  for $n = 3$. Consider a student with the scores of assignment (x_1), midterm (x_2), and final exam (x_3) given by $x_1 = 90$, $x_2 = 90$, and $x_3 = 46$. Then, his or her final grade point is given by

  $$\sum_{i=1}^{n} w_ix_i = w_1x_1 + w_2x_2 + w_3x_3 = 0.1 \times 90 + 0.3 \times 90 + 0.6 \times 46 = 63.6.$$

- We may compute the sum $\sum_{i=1}^n x_i$, the average $\bar{x} = (1/n)\sum_{i=1}^n x_i$, and the sample variance $(1/(n-1))\sum_{i=1}^n (x_i - \bar{x})^2$ when $n = 5$ and $x_i = i$, i.e., $x_1 = 1$, $x_2 = 2$, ..., $x_5 = 5$ as follows.
    - Compute $\sum_{i=1}^n x_i = \sum_{i=1}^5 i = 1 + 2 + 3 + 4 + 5 = 15$
    - Compute $\bar{x} = (1/n)\sum_{i=1}^n x_i = (1/5)\sum_{i=1}^5 i = (1/5)(1 + 2 + 3 + 4 + 5) = 3$
    - Compute $s^2 = (1/(n-1))\sum_{i=1}^n (x_i - \bar{x})^2 = (1/4)((1-\bar{x})^2 + (2-\bar{x})^2 + (3-\bar{x})^2 + (4-\bar{x})^2 + (5-\bar{x})^2) = (1/4)((-2)^2 + (-1)^2 + 0^2 + (1)^2 + (2)^2) = (1/4)(4+1+0+1+4) = 10/4 = 2.5$
    - We may show that $\sum_{i=1}^n (x_i - \bar{x}) = 0$ as follows.

$$
\begin{aligned}
\sum_{i=1}^5 (x_i - \bar{x}) &= (x_1 - \bar{x}) + (x_2 - \bar{x}) + (x_3 - \bar{x}) + (x_4 - \bar{x}) + (x_5 - \bar{x}) \\
&= (x_1 + x_2 + x_3 + x_4 + x_5) - 5\bar{x} \\
&= (x_1 + x_2 + x_3 + x_4 + x_5) - 5 \times (1/5)\sum_{i=1}^5 x_i \qquad \left(\text{because } \bar{x} = (1/5)\sum_{i=1}^5 x_i\right) \\
&= (x_1 + x_2 + x_3 + x_4 + x_5) - \sum_{i=1}^5 x_i \\
&= (x_1 + x_2 + x_3 + x_4 + x_5) - (x_1 + x_2 + x_3 + x_4 + x_5) \\
&= 0.
\end{aligned}
$$