Econ 325: Worksheet for Proofs[1]

Name and Student No. ____________________

Notations:

- The joint probability mass function is given by $p_{ij}^{X,Y} = P\left(X = x_i, Y = y_j\right)$ for $i = 1, \ldots n; j = 1, \ldots, m$.
- The marginal probability mass functions of X and Y are $p_i^X = P\left(X = x_i\right) = \sum_{j=1}^{m} p_{ij}^{X,Y}$ for $i = 1, \ldots n$ and $p_j^Y = P\left(Y = y_j\right) = \sum_{i=1}^{n} p_{ij}^{X,Y}$ for $j = 1, \ldots m$.

Question 1 *Prove that, for any constant a, b, and for any function $g(x)$,*

$$E[a + bg(X)] = a + bE[g(X)].$$

Proof:

$E[a+bg(X)] \overset{\text{def}}{=}$

Question 2 *Prove that, for any constant a, b, and for any function $g(x)$,*

$$Var[a + bX] = b^2 Var[X].$$

Proof:

$Var[a+bX] \overset{\text{def}}{=}$

Question 3 *Prove that, for any constant a, b, c,* $E[a + bX + cY] = a + bE[X] + E[Y]$.

Proof:

$$
\begin{aligned}
& E[a + bX + cY] \\
& \overset{\text{def}}{=} \sum_{i=1}^{n}\sum_{j=1}^{m}(a + bx_i + cy_j)p_{ij}^{X,Y} \\
& = \sum_{i=1}^{n}\sum_{j=1}^{m} ap_{ij}^{X,Y} + \sum_{i=1}^{n}\sum_{j=1}^{m} bx_i p_{ij}^{X,Y} + \sum_{i=1}^{n}\sum_{j=1}^{m} cy_j p_{ij}^{X,Y} \\
& = a\underbrace{\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{X,Y}}_{=1} + b\sum_{i=1}^{n} x_i \underbrace{\sum_{j=1}^{m} p_{ij}^{X,Y}}_{=?} + c\sum_{j=1}^{m} y_j \underbrace{\sum_{i=1}^{n} p_{ij}^{X,Y}}_{=?} \quad \text{(can you explain why this holds?)} \\
& =
\end{aligned}
$$

Question 4 *Prove that, when X and Y are independent,*

$$Cov\,(g(X), h(Y)) = 0$$

for any function $g(x)$ *and* $h(y)$.

Proof: Note that $P(X = x_i, Y = y_j) = P(X = x_i)P(Y = y_j)$ so that $p_{ij}^{X,Y} = p_i^X \times p_j^Y$ when X and Y are independent.

$$
\begin{aligned}
Cov(X, Y) & \overset{\text{def}}{=} E[(g(X) - E(g(X)))(g(Y) - E(g(Y)))] \\
& = \sum_{i=1}^{n}\sum_{j=1}^{m}[g(x_i) - E(g(X))][h(y_j) - E(h(Y))] \cdot \underbrace{p_{ij}^{X,Y}}_{=?} \\
& =
\end{aligned}
$$

Question 5 *Prove that, for any constant a and b,*

$$Var[aX + bY] = a^2Var[X] + b^2Var[Y] + 2abCov[X, Y].$$

Proof: Let $\tilde{X} \stackrel{\text{def}}{=} X - E(X)$ and $\tilde{Y} \stackrel{\text{def}}{=} Y - E(Y)$.

$$\begin{aligned}
Var[aX + bY] &\stackrel{\text{def}}{=} E[(aX + bY - E(aX + bY))^2] \\
&= E[(a(X - E(X)) + b(Y - E(Y)))^2] \\
&= E[(a\tilde{X} + b\tilde{Y})^2] \qquad \text{(because } \tilde{X} \stackrel{\text{def}}{=} X - E(X) \text{ and } \tilde{Y} \stackrel{\text{def}}{=} Y - E(Y)\text{)} \\
&=
\end{aligned}$$

Question 6 *Prove that, if X is a Bernoulli random variable with $P(X = 1) = p$, then $E[X] = p$.*

Proof: $E[X] \stackrel{\text{def}}{=} \sum_{x=0,1} xP(X = x) = 0 \times (1 - p) + 1 \times p =$

Question 7 *Prove that, if X and Y are two Bernoulli random variables with $P(X = 1) = P(Y = 1) = p$, then $E[X + Y] = 2p$.*

Proof: We can use the result of Question 3.

Question 8 *Prove that, if X is a Bernoulli random variable with $P(X = 1) = p$, then $Var[X] = p(1 - p)$.*

Proof:

$$\begin{aligned} Var[X] &\stackrel{\text{def}}{=} \sum_{x=0,1} (x - E[X])^2 P(X = x) \\ &= (0 - \underbrace{E[X]}_{=?})^2 (1 - p) + (1 - E[X])^2 p \\ &= \end{aligned}$$

Question 9 *Prove that, if X and Y are two Bernoulli random variables that are stochastically independent with $P(X = 1) = P(Y = 1) = p$, then $Var[(X + Y)/2] = \frac{p(1-p)}{2}$.*

Proof: We can use the result of Questions 2 and 5.

$$\begin{aligned} Var\left[\frac{X + Y}{2}\right] &= \left(\frac{1}{2}\right)^2 Var[X + Y] \quad \text{(by the result of Question 2)} \\ &= \end{aligned}$$

Question 10 *Prove that, if X_i for $i = 1, ..., n$ are n Bernoulli random variables that are stochastically independent to each other with $P(X_i = 1)$ for $i = 1, ..., n$, then (1) $E[\bar{X}] = p$ and (2) $Var[\bar{X}] = \frac{p(1-p)}{n}$.*

Proof: