FUNCTIONS

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1. DEFINITION AND EXAMPLES

We have already used functions in this course, so perhaps we should have defined them earlier. Anyway, a **function** from a set A to a set B is a rule that assigns to each $a \in A$ one and only one $b \in B$. If we want to call this function f, we denote this by $f : A \to B$, which is read as "f is a function from A to B" or simply "f from A to B." The set A is called the **domain** of f. B is called the **target** of f. The set

$$\{y \in B : f(x) = y \text{ for some } x \in A\}$$

is called the **image** of f.

Example 1.1.

(1) Production functions: $f : \mathbb{R}^2 \to \mathbb{R}$
- Linear $f(x_1, x_2) = a_1 x_1 + a_2 x_2$
- Cobb-Douglas: $f(x_1, x_2) = K x_1^{\alpha_1} x_2^{\alpha_2}$
- Constant elasticity of substitution: $f(x_1, x_2) = K(c_1 x_1^{-a} + c_2 x_2^{-a})^{-b/a}$

(2) Utility functions: $u : \mathbb{R}^T \to \mathbb{R}$
- Constant relative risk aversion: $u(c_1, ..., c_T) = \sum_{t=1}^T \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$
- Constant absolute risk aversion: $u(c_1, ..., c_T) = \sum_{t=1}^T \beta^t (-e^{-\alpha c_t})$

(3) Demand function with constant elasticity, $D : \mathbb{R}^3 \to \mathbb{R}^2$

$$D(p_1, p_2, y) = \begin{pmatrix} M p_1^{\alpha_{11}} p_2^{\alpha_{12}} y^{\beta_1} \\ M p_1^{\alpha_{21}} p_2^{\alpha_{22}} y^{\beta_2} \end{pmatrix}$$

where p_1 and p_2 are the prices of two goods and y is income.

I do not expect you to remember the names of these functions, but it is very likely that you will repeatedly encounter them this year.

1.1. **Visualizing functions.**
It is often useful to visualize a function. Simon and Blume have a whole section (13.2) about how to draw graphs of functions. That may have made sense in 1994, but it seems excessive now. To graph a function, use a computer. Wolfram alpha is a pretty good website for creating a quick graph. I'm sure you can find many other websites and cell phone apps with similar plotting capabilities. You can create nicer graphs using something like R or Matlab (or probably excel, or python, or whatever). You are probably familiar with indifference curves and isoquants from another economics course. Indifference curves and isoquants are examples of level sets.

FIGURE 1. CES, $a = 2$, $b = \frac{4}{5}$

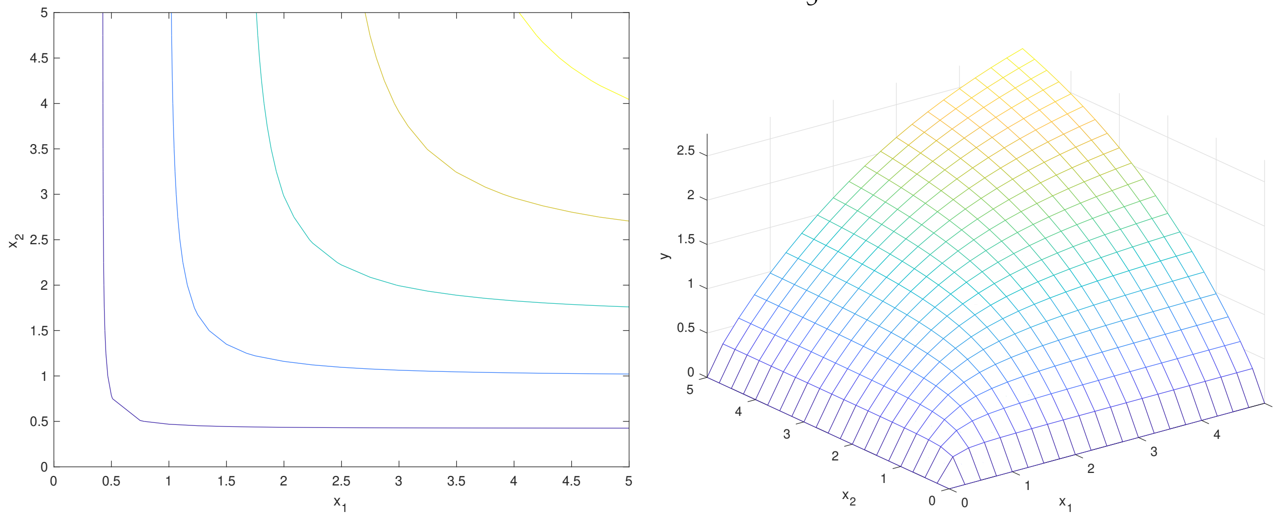
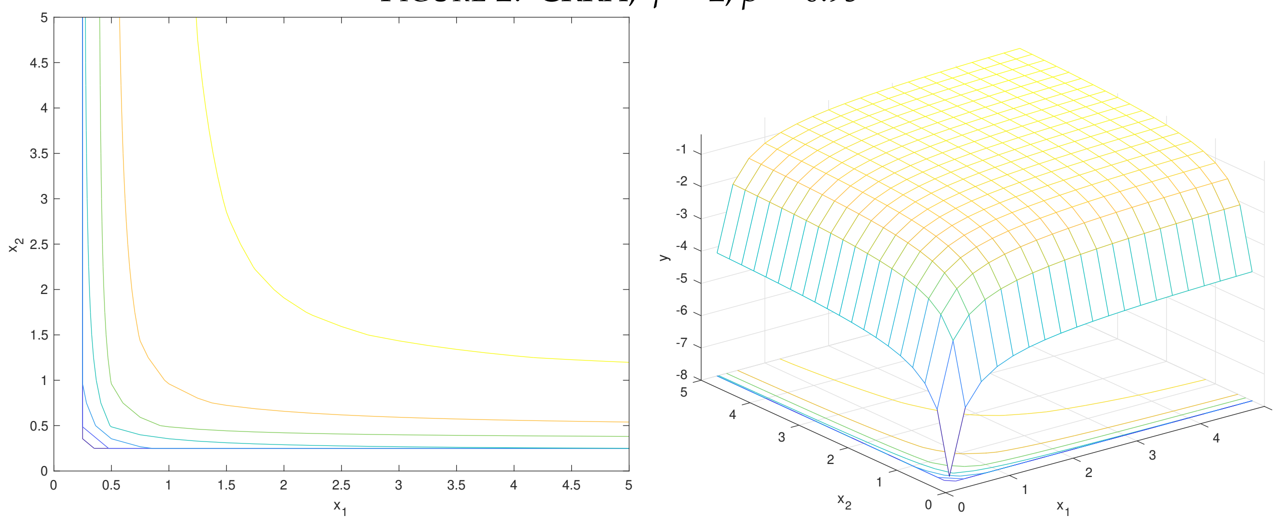


FIGURE 2. CRRA, $\gamma = 2$, $\beta = 0.95$



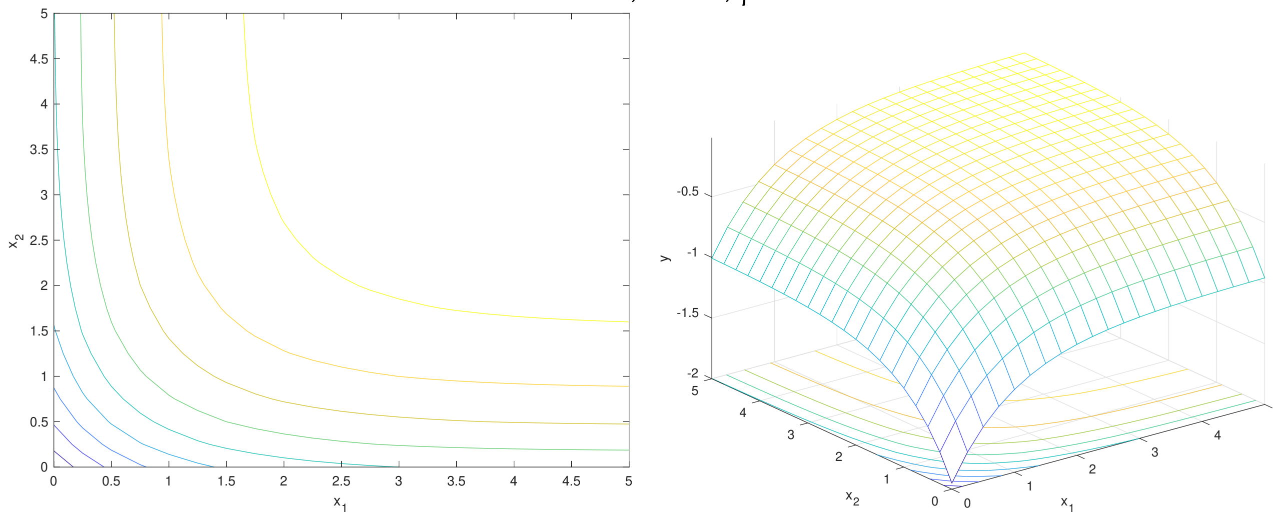
Definition 1.1. The **level sets** of a function $f : X \to Y$ are sets of the form

$$\{x \in X : f(x) = y\}$$

for some fixed $y \in Y$.

When you draw indifference curves, you are drawing level sets of a utility function. When you draw isoquants, you are drawing level sets of a production function. Figures 1-3 show isoquants and indifference curves for some of the examples of functions above.

FIGURE 3. CARA, $\alpha = 1$, $\beta = 0.95$



2. SPECIAL TYPES OF FUNCTIONS

There are some special types of functions that you are probably familiar with on $\mathbb{R}^1$ that we will generalize to $\mathbb{R}^n$. We have already covered linear functions in detail, although we called them linear transformations.

Definition 2.1. A function $f : V \to W$ where V and W are vector spaces is **linear** if f preserves addition and scalar multiplication, ie

- $f(x + y) = f(x) + f(y)$
- $f(\alpha x) = \alpha f(x)$

As we have already seen, linear functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be represented by m by n matrices. We went over a, perhaps very confusing, proof of this fact. You can find the same result in theorem 13.2 of Simon and Blume. Perhaps that proof will be clearer if you still find the relationship between matrices and linear functions confusing.

You probably are familiar with quadratic functions from $\mathbb{R}$ to $\mathbb{R}$. They look like

$$a_0 + a_1 x + a_2 x^2.$$

We can generalize this to functions from $\mathbb{R}^n$ to $\mathbb{R}$ as follows.

Definition 2.2. $q : \mathbb{R}^n \to R$ is a **quadratic** if

$$q(x_1, ..., x_n) = a_0 + \sum_{i=1, j \geq i}^{n} a_{ij} x_i x_j$$

A quadratic function can be written using matrix notation as

$$q(x_1, ..., x_n) = a_0 + x^T A x$$

where $A = \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} & \cdots & \frac{1}{2}a_{1n} \\ \frac{1}{2}a_{12} & a_{22} \cdots & \frac{1}{2}a_{2n} & \\ \vdots & \ddots & \ddots & \vdots \\ \frac{1}{2}a_{1n} & \cdots & \cdots & a_{nn} \end{pmatrix}$. Note that this choice of A is not unique. The $\frac{1}{2}$'s below the diagonal and the $\frac{1}{2}$'s above the diagonal can be replaced by any two numbers whose sum is 1. If you want to practice matrix multiplication, you could verify this. It might be easiest to start with a 2 by 2 or 3 by 3 example.

Next, we generalize polynomials to $\mathbb{R}^n$.

Definition 2.3. A **monomial** $f : \mathbb{R}^n \to R$ is any function of the form

$$f(x_1, ..., x_n) = c x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$$

where a_i are nonnegative integers.

$\sum_{i=1}^n a_i$ is the **degree** of the monomial.

A **polynomial** $f : \mathbb{R}^n \to \mathbb{R}$ is the sum of finitely many monomials, i.e.

$$f(x_1, ..., x_n) = \sum_{k=1}^{k} c_k x_1^{a_{1k}} \cdots x_n^{a_{nk}}$$

The maximum degree of the monomials making up a polynomial is the degree of the polynomial.

A useful property of linear functions is that for scalars t, $f(tx) = tf(x)$. Functions can have this sort of property even why they are not linear. For example, a Cobb-Douglas production function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x) = x_1^\alpha x_2^{1-\alpha}$$

with $\alpha \in [0, 1]$ satisfies

$$f(tx) = (tx_1)^\alpha (tx_2)^{1-\alpha} = t x_1^\alpha x_2^{1-\alpha} = tf(x).$$

Definition 2.4. A function $f : V \to W$ which V and W are real vector spaces is **homogenous of degree k** if

$$f(tx) = t^k f(x)$$

for all $x \in V$, $t \in \mathbb{R}$.

Example 2.1. Linear functions are homogenous of degree 1.

Example 2.2. A production function that is homogenous of degree 1 has constant returns to scale because doubling each of the inputs doubles the output. A production function that is homogenous of degree less than 1 has decreasing returns to scale. A production of that is homogenous of degree greater than 1 has increasing returns to scale.

Functions need not be homogenous at all.

Example 2.3. An affine transformation, $f(x) = Ax + b$, is not homogenous if $b \neq 0$.

In economics, especially when working with utility functions, we do not care about the exact value of a function, but rather how it ranks various bundles of goods. If $u : X \to \mathbb{R}$, where X is a metric space, is a utility function, then x_1 is (weakly) preferred to x_2 if $u(x_1) \geq u(x_2)$. If we multiply u by a positive constant the preferences over goods do not change. Similarly, if replace u with $\exp(u(\cdot))$, the preferences over goods would not change.

Definition 2.5. Let $f : \mathbb{R} \to \mathbb{R}$. f is **strictly increasing** if for all $x_1 > x_2$, $f(x_1) > f(x_2)$.
f is **strictly decreasing** if for all $x_1 > x_2$, $f(x_1) < f(x_2)$.
f is **strictly monontonic** if it is either strictly increasing or decreasing.
If the strict inequalities ($<$ and $>$) are replaced with weak inequalities ($\leq$ and $\geq$), then we would say f is **weakly** increasing / decreasing / monotonic.

Multiplying by a positive constant and exponentiation are both examples of strictly increasing functions. Monotonic functions can transform a homogenous function into a non-homogenous function. For example the identity function, $I : \mathbb{R} \to \mathbb{R}$ defined by $I(x) = x$ is homogenous of degree one, but $\exp(x)$ is not homogenous, nor is $x + 1$. When dealing with utility functions it is useful to recognize when a function is a monotonic transformation away from being homogenous.

Definition 2.6. Let $f : V \to \mathbb{R}$ where V is a vector space. f is **homothetic** if $\exists$ a homogenous $g : V \to \mathbb{R}$ and a monotonic $h : \mathbb{R} \to \mathbb{R}$ asuch that $h \circ g : V \to R$ defined by $(h \circ g)(x) = h(g(x))$ is equal to f.

3. Continuous functions

A continuous function is a function without any jumps or holes. Formally,

Definition 3.1. A function $f : X \to Y$ where X and Y are metric spaces is **continuous** at x if whenever $\{x_n\}_{n=1}^{\infty}$ converges to x in X, then $f(x_n) \to f(x)$ in Y.

We simply say that f is continuous if it is continuous at every $x \in X$. There are some equivalent definitions of continuity that are also useful. You may have seen continuity defined as the result of the following lemma.

Lemma 3.1. *$f : X \to Y$ is continuous at x if and only if for every $\epsilon > 0$ $\exists$ $\delta > 0$ such that $d(x, x') < \delta$ implies $d(f(x), f(x')) < \epsilon$.*

Proof. On problem set.

□

A third way of defining continuity is in terms of open sets. First, another definition.

Definition 3.2. Let $f : X \to Y$. The **preimage** of $V \subseteq Y$ is the set in X, $f^{-1}(V)$ defined by

$$f^{-1}(V) = \{x \in X : f(x) \in V\}$$

A function is continuous if and only if the preimage of any open set is open.

Lemma 3.2. *$f : X \to Y$ is continuous if and only if $f^{-1}(V)$ is open for all open $V \subseteq Y$.*

Proof. Suppose for all open $V \subseteq Y$ that $f^{-1}(V)$ is also open. We want to show that then f is continuous. To do that, let $x_n \to x$ and let $\epsilon > 0$. $N_\epsilon(f(x))$ is open, so by assumption, $f^{-1}(N_\epsilon(f(x)))$ is also open. By the definition of open sets, $\exists\ \delta > 0$ such that $N_\delta(x) \subseteq f^{-1}(N_\epsilon(f(x)))$. By the definition of $x_n \to x$, $\exists N$ such that if $n \geq N$, $x_n \in N_\delta(x)$. Then $x_n \in f^{-1}(N_\epsilon(f(x)))$, so $f(x_n) \in N_\epsilon(f(x))$, i.e.

$$d\left(f(x_n), f(x)\right) < \epsilon.$$

Therefore, $f(x_n) \to f(x)$.

Conversely, suppose f is continuous. Let $V \subseteq Y$ be open. We want to show that $f^{-1}(V)$ is also open. Suppose it is not open. Then $\exists x \in f^{-1}(V)$ such that for any $\epsilon > 0$, $\exists \tilde{x}_\epsilon \notin f^{-1}(V)$ with

$$d(x, \tilde{x}_\epsilon) < \epsilon.$$

Pick a sequence of ϵ_n that converges to zero, such as $\epsilon_n = 1/n$. Then the associated $\tilde{x}_n \to x$. However, since each $\tilde{x}_n \notin f^{-1}(V)$, $f(\tilde{x}_n) \in V^c$. But then having $f(\tilde{x}_n) \to f(x)$ would mean that V^c is not closed, which contradict V being open. Thus, $f^{-1}(V)$ must be open when f is continuous. □

Since the a set is open if and only its complement is closed, we can also define continuity using closed sets.

Corollary 3.1. *$f : X \to Y$ is continuous if and only if $f^{-1}(V)$ is closed for all closed $V \subseteq Y$.*

Proof. Let $V \subseteq Y$ be closed. Then V^c is open. Also, note that the complement of the preimage of V is the preimage of V^c. In symbols,

$$f^{-1}(V)^c = \{x \in X : f(x) \notin V\} = \{x \in X : f(x) \in V^c\} = f^{-1}(V^c).$$

From lemma 3.2, f is continuous iff $f^{-1}(V^c) = f^{-1}(V)^c$ is open for all open sets V^c, which is true iff $f^{-1}(V)$ is closed for all closed sets V. □

Earlier we saw that convergence of sequences is preserved by arithmetic. Since continuity can be defined using sequences, it should be no surprise that continuity is also preserved by arithmetic.

Theorem 3.1. *Let $f : X \to Y$ and $g : X \to Y$ be continuous and X and Y be vector spaces. Then $(f + g)(x) = f(x) + g(x)$ is continuous.*

Proof. If f and g are continuous, then by definition $f(x_n) \to f(x)$ and $g(x_n) \to g(x)$ whenever $x_n \to x$. From the previous lecture the limit of a (finite) sum is the sum of limits, so $f(x_n) + g(x_n) \to f(x) + g(x)$, and $f + g$ is continuous. □

Similar results can be shown for subtraction, multiplication, etc, whenever they are well defined.

Continuity is also preserved by composition.

Theorem 3.2. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous where X, Y, and Z are metric spaces. Then $f \circ g$ is continuous, where*

$$(f \circ g)(x) = f(g(x)).$$

Proof. Let $x_n \to x$. g is continuous, so $g(x_n) \to g(x)$. f is also continuous, so $f(g(x_n)) \to f(g(x))$. □

$f \circ g$ is called the composition of f and g.

3.1. **Onto, one-to-one, and inverses.** We have already used the concepts of onto, one-to-one, and inverses. We restates the definitions here for completeness.

Definition 3.3. $f : X \to Y$ is **one-to-one** or **injective** if for all $x_1, x_2 \in X$,

$$f(x_1) = f(x_2)$$

if and only if $x_1 = x_2$.

Equivalently, f is injective if for each $y \in Y$, the set $\{x : f(x) = y\}$ is either a singleton or empty. In terms of a nonlinear equation, if f is one-to-one, then $f(x) = b$ has at most one solution.

Definition 3.4. $f : X \to Y$ is **onto** or **surjective** if $\forall y \in Y$, $\exists x \in X$ such that $f(x) = y$.

In terms of a nonlinear equation, if f is onto, then $f(x) = b$ has at least one solution. When f is one-to-one and onto, we say that f is **bijective**. A bijective function has an inverse.

Definition 3.5. If $f : X \to Y$ is bijective, then the **inverse** of f, written f^{-1} satisfies

$$f(f^{-1}(y)) = y$$

and

$$f^{-1}(f(x)) = x.$$

Comment 3.1. While writing these notes, I briefly tried to prove that if $f : X \to Y$ is bijective and continuous, then f^{-1} is continuous. I could not do this, which is good, because that statement is false. You have to be a little creative in defining X and Y to come up with a counterexample. Let $X = [0, 2\pi)$ and $Y = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Then $f(x) = (cos(x), sin(x))$ is bijective and continuous, but f^{-1} is not continuous at $(1, 0)$.

This counterexample is actually related to a fundamental fact in topology. You may remember from last lecture that topology is about studying spaces with open and closed sets that do not necessarily have a metric. One thing that people are interested when studying such spaces is finding a continuous (in both directions) bijections between them. Loosely speaking, two topological spaces will have a continuous bijection between them if one can be bent and stretched from one into the form of another. You cannot bend a circle into an interval without breaking the circle, so there is no continuous bijection between the circle and an interval. When there is a continuous bijection between two spaces, they have the same collection of open sets, so to a topologist, they are the same. We then call the spaces homeomorphic (or topologically isomorphic). Loosely speaking, spaces will be homeomorphic if they are the same dimension and their shapes have the same number of holes. The circle has one hole, an interval has none, so they are not topologically isomorphic. I'd be remiss not to make a joke now, so here goes: Why did the topologist eat her/his coffee mug and drink from his/her donut? Because they're topologically isomorphic. Hahaha.

4. CORRESPONDENCES

[1]A function, $f : X \to Y$ associates exactly one element of Y, $f(x)$, with each $x \in X$. Often we encounter things that are like functions, but for each $x \in X$, there are multiple elements of Y. We call this generalization of a function as correspondence.

Definition 4.1. A **correspondence** from a set X to a set Y, is a rule that assigns to each a $x \in X$ a subset of Y. We denote a correspondence by $\phi : X \rightrightarrows Y$.

An equivalent definition is that $\phi : X \rightrightarrows Y$ is a function from X to the power set of Y. Correspondences appear often in economics, especially as constraint sets in optimization problems.

Example 4.1 (Budget correspondence)**.** Suppose there are n goods with prices $p \in \mathbb{R}^n$. Then given income of m, a consumer can afford $\chi(p, m) = \{x \in X \subseteq \mathbb{R}^n : p'x \leq m\}$, which defines a correspondence $\chi : \mathbb{R}^{n+1} \rightrightarrows X$. We can write the consumer's problem of maximizing utility subject to the budget constraint as

$$\max_{x \in \chi(p,m)} u(x)$$

If this problem has a solution, then the indirect utility function is the maximized utility,

$$v(p, m) = \max_{x \in \chi(p,m)} u(x).$$

The demand correspondence (usually function) is

$$x^*(p, m) = \arg\max_{x \in \chi(p,m)} u(x).$$

Such maximization problems are central to economics. To derive properties of the indirect utility and demand functions it is often useful to treat the budge set as a correspondence.

Correspondences also appear in economics in any model where we multiple equilibria, such as many games.

Defining continuity is a bit more complicated for correspondences than for functions. A function can either be continuous or it can jump. A correspondence can also expand or contract. For example, consider $\xi : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$\xi(x) = \begin{cases} [0, 1] & \text{if } x > 0 \\ [1/4, 3/4] & \text{if } x \leq 0 \end{cases}$$

and $\psi : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$\psi(x) = \begin{cases} [0, 1] & \text{if } x \geq 0 \\ [1/4, 3/4] & \text{if } x < 0 \end{cases}$$

Both these correspondences are somewhat continuous because they contain a continuous function, e.g. $f(x) = 1/2$, for all x. However, they are also somewhat discontinuous because the corresponding set changes suddenly at 0. Motivated by this observation we define the following:

[1]This section is largely based on section 2.1.5 of Carter.

Definition 4.2. A correspondence, $\phi : X \rightrightarrows Y$ is **upper hemicontinuous** at x if for all sequences $x_n \to x$ and $y_n \in \phi(x_n)$ with $y_n \to y$, then $y \in \phi(x)$.

In the previous example, ψ is upper hemicontinuous at 0, but ξ is not. To see this consider $x_n = 1/n$ and $y_n = 1$.

Definition 4.3. A correspondence, $\phi : X \rightrightarrows Y$ is **lower hemicontinuous** at x if for all sequences $x_n \to x$ and $y \in \phi(x)$, there exists a subsequence, x_{nk} and $y_k \in \phi(x_{nk})$ with $y_k \to y$.

In the previous example, ξ is lower hemicontinuous at 0, but ψ is not. To see this consider $x_n = -1/n$ and $y = 1$.

Definition 4.4. We say that a correspondence is **continuous** if it is both upper and lower hemicontinuous.

At all $x \neq 0$, ξ and ψ are continuous.