CHAPTER 1: ELEMENTARY CALCULUS

W. Erwin Diewert May 2003.

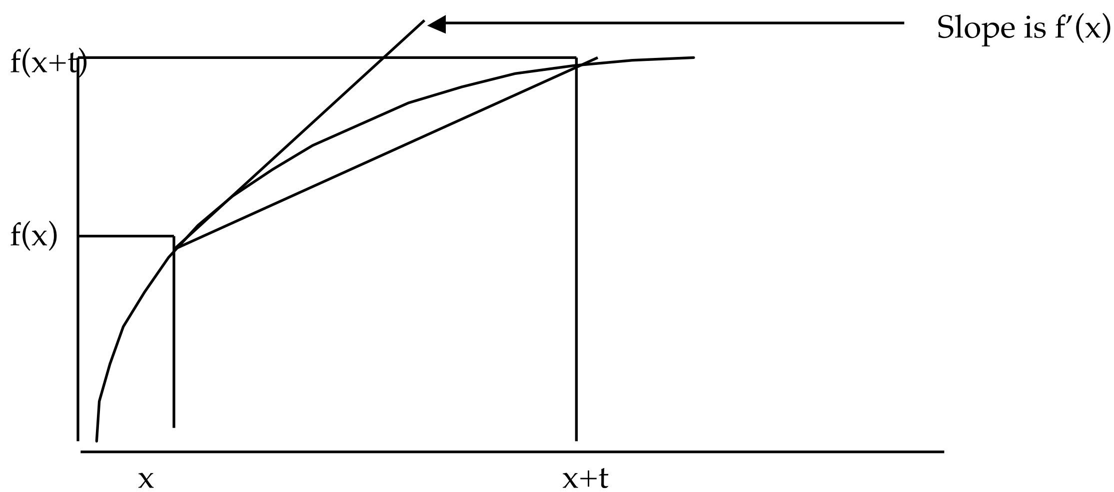
1. The Derivative of a Function of One Variable.

Let f(x) be a function of one variable x.

Definition 1: The derivative of f evaluated at x is the following limit (if the limit exists):

(1) $$\lim_{t \to 0} [f(x + t) - f(x)]/t \equiv f'(x) \text{ or } \equiv df(x)/dx.$$

Geometrically, $f'(x)$ may be interpreted as the slope of the line tangent to the graph of the function through the point x, f(x) if such a tangent line exists.



As t approaches 0, the slopes of the line segments connecting (x,f(x)) to (x+t,f(x+t)) get closer to $f'(x)$.

Rules For Differentiation

Function	Derivative
Rule 1: for $k \neq 0$	$f(x) = ax^k$, $f'(x) = kax^{k-1}$
Rule 2:	$f(x) = e^x$, $f'(x) = e^x$
Rule 3: for $x > 0$	$f(x) = \ln x$, $f'(x) = 1/x$
Rule 4: *Constant Rule*	$f(x) = kg(x)$, $f'(x) = kg'(x)$
Rule 5: *Addition Rule*	$f(x) = g(x) + h(x)$, $f'(x) = g'(x) + h'(x)$

Rule 6: $f(x) = g(x)h(x)$, $f'(x) = g'(x)h(x) + g(x)h'(x)$ *Product Rule*
Rule 7: $f(x) = g[h(x)]$, $f'(x) = g'[h(x)]h'(x)$ *Chain Rule*
Rule 8: $f(x) = g(x)/h(x)$, $h(x) \neq 0$
$f'(x) = \{g'(x)h(x) - g(x)h'(x)\}/h(x)^2$ *Quotient Rule*

Examples

(1) $f(x) \equiv a + bx + cx^2 + dx^3$, $f'(x) = b + 2cx + 3dx^2$.

(2) $f(x) \equiv e^{kx} \equiv e^{h(x)}$ where $h(x) \equiv kx$
$\equiv g[h(x)]$ where $g(y) \equiv e^y$

Therefore

$$\begin{aligned} f'(x) &= g'[h(x)]h'(x) \\ &= e^{h(x)}\, h'(x) \\ &= e^{kx}k = ke^{kx}. \end{aligned}$$

(3) $f(x) \equiv [g(x) + h(x)]^k$, $f'(x) = k[g(x) + h(x)]^{k-1}[g'(x) + h'(x)]$.

Higher Order Derivatives

In example (1) above, $f'(x)$ is a differentiable function of x for each x. Hence we may calculate the derivative of $f'(x)$. [Geometric interpretation?]

Definition 2: The second derivative of f evaluated at x is (assuming that $f'(x)$ exists in a neighbourhood around x and that the limit below exists):

(2) $f''(x) \equiv \lim_{t \to 0}[f'(x + t) - f'(x)]/t$
$\equiv d^2f(x)/dx^2$ (alternative notation).

Examples. Calculate $f''(x)$ for examples (1) and (2) above.

(1) $f'(x) = b + 2xc + 3dx^2$, $f''(x) = 2c + 6dx$

(2) $f'(x) = ke^{kx}$, $f''(x) = k\{de^{kx}/dx\}$
$= k\,\{ke^{kx}\}$
$= k^2e^{kx}$.

The third derivative of f evaluated at x is defined as the derivative of the function $f''(x)$ if it exists. It is denoted by $f'''(x)$ or $d^3f(x)/dx^3$.

2. Maximizing or Minimizing Differentiable Functions of One Variable

Consider the problem of maximizing or minimizing the function of one variable, $f(x)$, over the set $x > 0$. Assume $f'(x)$ exists for $x > 0$.

If we are at a local maximum of f at the point x^*, then we must be at the top of a hill and the slope of the function must be zero there; i.e., we must have $f'(x^*) = 0$.

Similarly, if f attains a local minimum at the point x^*, then we must be at the bottom of a valley and the slope must be zero there; i.e., we must have $f'(x^*) = 0$.

Thus we have:

The First Order Necessary Condition for f to attain a local max or min at the interior point x^* is:

(1) $f'(x^*) = 0$.

Note that condition (1) above is only a necessary condition for a local min or max; i.e., if f attains a local min or max at x^*, then (1) will be satisfied. However, if condition (1) is satisfied, then we do not know whether f attains a local min or a local max at x^*. In fact, f might not even attain a local min or max at x^*; i.e., condition (1) could be satisfied at a point of inflection. For example, let $f(x) \equiv (x - 1)^3$. Then $f'(x) = 3(x - 1)^2 (1 + 0)$ so that $f'(1) = 0$. However, $x^* = 1$ is not a local maximizer or minimizer of f.

Second Order Sufficient Conditions for f to attain a *strict local maximum* at the interior point x^*:

(2) $f'(x^*) = 0$ and $f''(x^*) < 0$.

The second part of (2) means that the rate of change of $f'(x)$ around x^* is negative. This means that $f'(x)$ is positive for x slightly less than x^*, $f'(x) = 0$ for $x = x^*$, and $f'(x)$ is negative for x slightly greater than x^*.

Example 1:

$f(x) \equiv 2 - x^2$ for $-\infty < x < +\infty$.

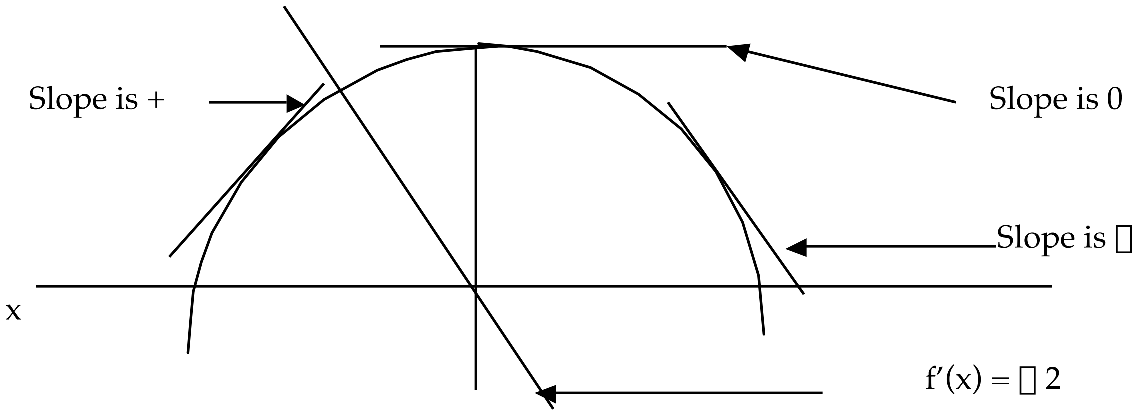
$f(x) = 0 - 2x \stackrel{set}{=} 0$

$\Rightarrow -2x = 0$

$\Rightarrow x^* = 0$

$f''(x) = -2 = f''(0)$.

Therefore f attains a local maximum at $x^* = 0$. Since there are no other points where the slope is 0, $x^* = 0$ is the global maximizer of f.



Second Order Sufficient Conditions for f to attain a *strict local minimum* at the interior point x*:

(3) $f'(x^*) = 0$ and $f''(x^*) > 0$.

If conditions (3) are satisfied, the slope of f is negative to the left of x*, 0 at x*, and positive immediately to the right of x*.

Example 2:

$$f(x) \equiv x^2 - 1 \qquad \text{for } -\infty < x < +\infty .$$
$$f'(x) = 2x + 0 \overset{set}{=} 0 \qquad \Rightarrow x^* = 0$$
$$f''(x) = 2 \qquad \Rightarrow f''(x^*) = 2 > 0.$$

Therefore x* = 0 is a local minimizer. It is also the global minimizer of f (same reason as example 1).

Example 3:

$$f(x) \equiv x^3 - x^2 + 2 \qquad -\infty < x < +\infty .$$
$$f'(x) = 3x^2 - 2x \overset{set}{=} 0$$
$$\text{or} \quad x(3x - 2) = 0$$
$$\Rightarrow \quad x_1^* = 0,\ x_2^* = 2/3$$
$$f''(x) = 6x - 2$$
$$f''(x_1^*) = 6(0) - 2 = -2 \Rightarrow \text{local } \textit{max} \text{ at } x = 0$$
$$f''(x_2^*) = 6(2/3) - 2 = 2 \Rightarrow \text{local } \textit{min} \text{ at } x = 2/3$$

However, the local max is not a global max and the local min is not a global min. (Why?)

Example 4

$$f(x) = x^3; \qquad -\infty < x < +\infty .$$
$$f'(x) = 3x^2 \overset{set}{=} 0$$
$$\Rightarrow \quad x^* = 0$$
$$f''(x) = 6x$$

Thus $f''(0) = 0$.

The sufficient conditions for a local max or a local min are not satisfied at $x^* = 0$. Hence we cannot say if f attains a local max or min at $x^* = 0$.

Sufficient Conditions for f to have an *inflection* point at x^*.

(4) $f'(x^*) = 0$, $f''(x^*) = 0$ and $f'''(x^*) \neq 0$.

Thus the f defined in example 4 above has an inflection point at $x^* = 0$.

Second Order Necessary Conditions for f to attain a *local maximum* at the interior point x^*:

(5) $f'(x^*)= 0$ and $f''(x^*) \leq 0$.

Second Order Necessary Conditions for f to attain a *local minimum* at the interior point x^*:

(6) $f'(x^*) = 0$ and $f''(x^*) \geq 0$.

Conditions (4) - (6) are not as important as conditions (2) and (3).

3. Some Consumer Theory Examples of Optimizing Behavior.

Example 1: We suppose that a consumer's preferences over nonnegative amounts of two goods, $x_1 \geq 0$, $x_2 \geq 0$, can be represented by means of the following Cobb-Douglas utility function:

(1) $u(x_1, x_2) \equiv x_1^{\alpha} x_2^{1-\alpha}$, $0 < \alpha < 1$

where α is a fixed number or parameter which characterizes the consumer's preferences.

We suppose that the price of good 1 is some number $p_1 > 0$ and the price of the second good is $p_2 > 0$. We suppose also that the consumer has the fixed amount of income $I > 0$ to spend on the two goods. The consumer's budget constraint is

(2) $p_1x_1 + p_2x_2 = I$.

We assume that the consumer attempts to maximize the utility function (1) subject to the budget constraint (2) and the nonnegativity constraints $x_1 \geq 0$ and $x_2 \geq 0$. Mathematically, we write this constrained maximization problem as follows:

(3) $\max_{x_1, x_2} \{x_1^{\alpha} x_2^{1-\alpha} : p_1x_1 + p_2x_2 = I; x_1 \geq 0; x_2 \geq 0\}$.

How can we solve this rather complex looking problem (3) using the material in the previous section? We temporarily ignore the nonnegativity restrictions $x_1 \ge 0$ and $x_2 \ge 0$ and we use the budget constraint (2) to solve for x_2 in terms of x_1:

(4) $\quad x_2 = (I - p_1x_1)/p_2$

Now substitute (4) into the utility function (1) and we have reduced the two variable constrained maximization problem (3) into the following single variable utility maximization problem:

(5) $\quad \max_{x_1}\ f(x_1) \equiv x_1^{\alpha}\,[(I - p_1x_1)/p_2]^{1-\alpha}; \qquad x_1 \ge 0.$

$f'(x_1) = \alpha x_1^{\alpha-1}\,[(I - p_1x_1)/p_2]^{1-\alpha} + x_1^{\alpha}(1-\alpha)\,[(I - p_1x_1)/p_2]^{1-\alpha-1}(-p_1/p_2) \overset{\text{set}}{=} 0$

or $\quad \alpha x_1^{\alpha-1} = x_1^{\alpha}(1 - \alpha)\,[(I - p_1x_1)/p_2]^{-1}(p_1/p_2)$

or $\quad \alpha x_1^{-1} = (1-\alpha)(I - p_1x_1)^{-1}p_1$

or $\quad \alpha(I - p_1x_1) = (1 - \alpha)p_1x_1$

or $\quad \alpha I = \alpha p_1x_1 + (1 - \alpha)p_1x_1 = p_1x_1$

(6) or $\quad x_1^* = \alpha I/p_1 > 0.$

It can be verified that $f''(\alpha I/p_1) < 0$ so that the x_1^* defined by (6) does in fact solve the maximization problem (5). We may substitute (6) into (4) and determine the corresponding x_2^* which solves (3):

(7) $\quad x_2^* = (I - p_1x_1^*)/p_2 = (I - p_1(\alpha I/p_1))/p_2 = (1 - \alpha)I/p_2 > 0.$

Since the x_1^* and x_2^* defined by (6) and (7) are both positive, the nonnegativity restrictions $x_1 \ge 0$ and $x_2 \ge 0$ are satisfied. Hence we can conclude that we have solved the consumer's constrained utility maximization problem (3). Note that the x_1 and x_2 solutions are functions of p_1, p_2 I and α; i.e., we have

(8) $\quad x_1^* = D_1(p_1, p_2, I, \alpha) \equiv \alpha I/p_1$ and

$\quad x_2^* = D_2(p_1, p_2, I, \alpha) \equiv (1 - \alpha)I/p_2.$

The solution functions D_1 and D_2 are the consumer's system of utility maximizing demand functions. Note that the functional forms for D_1 and D_2 are completely determined by the functional form for the consumer's utility function u; see (1).

This example illustrates how optimizing theory is used in modern microeconomic theory. It also illustrates a technique that we shall use quite frequently in dealing with constrained maximization problems: we shall use the constraint to solve for one variable in terms of the other variables and then reduce the constrained maximization problem into an unconstrained maximization problem involving one less variable.

Example 2: Notice that the parameter that characterizes the consumer's preferences, α, occurred in both of the demand functions (8) in the previous example. How can we determine what α is?

Suppose we collect data on the consumer's purchases of commodity 1 during period t, x_1^t say, on the market price for commodity 1 during period t, p_1^t say and on the consumer's income in period t, I^t say. Then if equation (6) held exactly in each period, we would have $x_1^t = \alpha I^t/p_1^t$ or $p_1^t x_1^t / I^t = \alpha$ for t = 1, 2, . . ., T. It is unlikely that equation (6) will hold exactly for each period. Thus we might have:

(9) $$p_1^t x_1^t / I^t = \alpha + e_t; \; t = 1, 2, \ldots, T,$$

where e_t is an *error term* for period t. It is reasonable to estimate or approximate the consumer's α parameter by choosing α to *minimize* the following function:

(10) $$f(\alpha) \equiv \Sigma_{t=1}^{T} e_t^2 = \Sigma_{t=1}^{T} [(p_1^t x_1^t / I^t) - \alpha]^2$$

= the sum of the squared errors.

$$f'(\alpha) = \Sigma_{t=1}^{T} 2[(p_1^t x_1^t / I^t) - \alpha]^{2-1}(-1) \overset{set}{=} 0$$

$$= -2\, \Sigma_{t=1}^{T}(p_1^t x_1^t / I^t) + 2\, \Sigma_{t=1}^{T} \alpha = 0$$

(11) $$\Rightarrow \alpha^* = [\Sigma_{t=1}^{T}(p_1^t x_1^t / I^t)]/T$$

$f''(\alpha^*) = 2T > 0$ so we have a local (and global) minimum of $f(\alpha)$ at α^*. The number α^* defined by (11) is called the *least squares estimator* for α.

Now you can see how we can use observable data on a consumer's choices in order to determine an approximation to his or her preference function. (Why are we interested in a consumer's preferences anyway?)

4. Partial Derivatives

Each consumer and producer in a modern economy must choose between thousands of goods. Hence, we cannot restrict ourselves to optimization problems involving only one variable. However, it turns out that the single variable optimization techniques studied in section 2 can be modified to deal with the N variable case. A key concept that we will need to accomplish this generalization to the N variable case is the idea of a partial derivative.

Let f be a function of N variables, $x_1, x_2, \ldots, x_N$.

Definition 1: The first order partial derivative of f with respect to x_1 evaluated at $x_1, x_2, \ldots, x_N$ is defined as the following limit (if it exists):

$$
\begin{aligned}
(11) \quad & \lim_{t \to 0}[f(x_1 + t, x_2, \ldots, x_N) - f(x_1, x_2, \ldots, x_N)]/t \\
& \equiv \partial f(x_1, x_2, \ldots, x_N)/\partial x_1 \\
\text{or} \quad & \equiv f_{x_1}(x_1, x_2, \ldots, x_N) \\
\text{or} \quad & \equiv f_1(x_1, x_2, \ldots, x_N).
\end{aligned}
$$

Note that there are three commonly used notational conventions used to denote the concept of a partial derivative. Note that $x_2, \ldots, x_N$ are held constant in definition (1). Thus if we regard f as just a function of x_1, then (1) reduces to $f'(x_1)$, the ordinary derivative of f with respect to x_1. Hence in order to actually calculate $f_1(x_1, x_2, \ldots, x_N)$, we need only treat $x_2, \ldots, x_N$ as constants and differentiate the resulting function of x_1 with respect to x_1. The other N-1 first order partial derivatives of f may be defined in an analogous manner.

Example 1:

$$f(x_1, x_2, x_3) \equiv a_0 + a_1x_1 + a_2x_2 + a_3x_3,$$
$$f_1(x_1, x_2, x_3) \equiv a_1$$
$$f_2(x_1, x_2, x_3) = a_2$$
$$f_3(x_1, x_2, x_3) = a_3.$$

Example 2:

$$f(x_1, x_2, x_3) \equiv 3x_1^2 + 2x_1x_2 + 6x_2^{\frac{1}{2}}x_3^{\frac{1}{2}} + e^{x_2} + \ln x_3,\ x_i > 0.$$
$$f_1(x_1, x_2, x_3) = 6x_1 + 2x_2$$
$$f_2(x_1, x_2, x_3) = 2x_1 + 3x_2^{-\frac{1}{2}}x_3^{\frac{1}{2}} + e^{x_2}$$
$$f_3(x_1, x_2, x_3) = 3x_2^{\frac{1}{2}}x_2^{-\frac{1}{2}} + x_3^{-1}.$$

Example 3:

$$f(x_1, x_2) \equiv a_0 + a_1x_1 + a_2x_2 + a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2$$
$$f_1(x_1, x_2) = a_1 + 2a_{11}x_1 + a_{12}x_2$$
$$f_2(x_1, x_2) = a_2 + a_{12}x_1 + 2a_{22}x_2$$

If you can differentiate functions of one variable, then you can partially differentiate. It's easy; just treat the "other" variables as constants.

Question: What is the geometric interpretation of a partial derivative?

Higher Order Partial Derivatives. Once we have calculated the function $\partial f(x_1, \ldots, x_N)/\partial x_1 = f_1(x_1, \ldots, x_N)$, we can partially differentiate the resulting function with respect to $x_1, x_2, \ldots$, or x_N. The resulting function is called a *second order partial derivative* of f with respect to the variable x_1 and x_i say, evaluated at $x_1, \ldots, x_N$:

(2) $$\lim_{t\to 0} \frac{f_1(x_1,\ldots,x_{i-1}, x_i + t, x_{i+1},\ldots,x_N) - f_1(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_N)}{t}$$
$$\equiv \partial^2 f(x_1, \ldots, x_N)/\partial x_1\, \partial x_i$$
$$\equiv f_{x_1x_i}(x_1, \ldots, x_N)$$
$$\equiv f_{1i}(x_1, \ldots, x_N).$$

There are three commonly used notations used to denote the concept of a second order partial derivative.

Example 3:

$$f_1(x_1, x_2) = a_1 + 2a_{11}x_1 + a_{12}x_2$$
$$f_{11}(x_1, x_2) = 2a_{11}$$
$$f_{12}(x_1, x_2) = a_{12}$$
$$f_2(x_1, x_2) = a_2 + a_{12}x_1 + 2a_{22}x_2$$
$$f_{21}(x_1, x_2) = a_{12}$$
$$f_{22}(x_1, x_2) = 2a_{22}$$

Note that $f_{12}(x_1, x_2) = f_{21}(x_1, x_2)$. We shall show later that this is generally the case.

Maximizing or Minimizing a Differentiable Function of N Variables

If we are at a local interior maximizing or minimizing point $x_1^*, \ldots x_N^*$ of f, then we are at the top of a hill or at the bottom of a valley with respect to each variable taken separately. Thus the following conditions must be satisfied:

First Order Necessary Conditions for an Interior Min or Max:

(3) $$\partial f(x_1^*, \ldots, x_N^*)/\partial x_1 = f_1(x_1^*, \ldots, x_N^*) = 0$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$\partial f(x_1^*, \ldots, x_N^*)/\partial x_N = f_N(x_1^*, \ldots, x_N^*) = 0.$$

Note that (3) is a system of N simultaneous equations in N unknowns, $x_1^*, \ldots, x_N^*$.

So far, the conditions for maximizing or minimizing a function of one variable have generalized to the N variable case in a relatively straightforward manner. However, developing second order sufficient conditions for maximizing or minimizing a function of N variables requires that we introduce the concept of the directional derivative. In order to operationalize this concept, we shall require a few mathematical results.

5. The Mean Value Theorem

The Mean Value Theorem: Let f(x) be a continuous function over the interval $a \le x \le b$ with $a < b$. Suppose $f'(x)$ exists over this interval. Then there exists an x^* such that

(1) $a < x^* < b$ and

(2) $f'(x^*) = [f(b) - f(a)]/[b - a]$.

Proof: Define the function g(x) for $a \le x \le b$ by

(3) $g(x) \equiv f(x) - x\,[f(b) - f(a)]/[b - a]$.

Let $x = a$ and $x = b$ and evaluate $g(a)$ and $g(b)$. We find that

(4) $g(a) = f(a) - a\,[f(b) - f(a)]/[b - a] = [bf(a) - af(b)]/[b - a]$ and

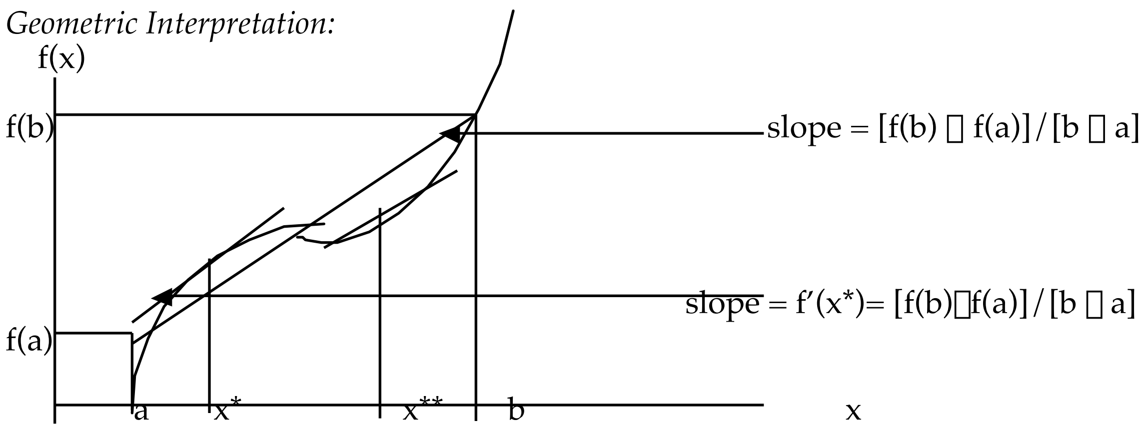
(5) $g(b) = f(b) - b\,[f(b) - f(a)]/[b - a]= [bf(a) - af(b)]/[b - a]$.

Since $g(a) = g(b)$ and g is continuous, g must attain either a local maximum or a local minimum (or both) at some point x^* between a and b. The first order necessary condition for g to attain a local max or min at x^* will be satisfied and thus we have:

(6) $0 = g'(x^*) = f'(x^*) - [f(b) - f(a)]/[b - a]$.

Now note that (2) is just a rearrangement of (6). Q.E.D.

Geometric Interpretation:



Note that $[f(b) - f(a)]/[b - a]$ is the average slope of the function over the interval $a \le x \le b$ while $f'(x^*)$ is the slope of the line tangent to the function at the point x^*.

6. A Multivariate Function Chain Rule

Theorem: Let $f(x_1, x_2)$ be a function of two variables defined over the region $a_1 < x_1 < b_1$ and $a_2 < x_2 < b_2$. Suppose that the first order partial derivatives of f exist and are continuous over this region. Suppose that $g_1(z)$ and $g_2(z)$ are differentiable functions of z for $c < z < d$ and for z in this region, we have $a_1 < g_1(z) < b_1$ and $a_2 < g_2(z) < b_2$. Finally, for $c < z < d$, define the (multivariate composite) function $h(z)$ by:

(1) $h(z) \equiv f[g_1(z), g_2(z)]$

Then the derivative of h is given by:

(2) $h'(z) = f_1[g_1(z), g_2(z)]g_1'(z) + f_2[g_1(z), g_2(z)]g_2'(z)$.

Proof: By the definition of a derivative, $h'(z)$ is defined as the following limit:

(3) $h'(z) \equiv \lim_{t \to 0} [f(g_1(z + t), g_2(z + t)] - f[g_1(z), g_2(z)]]/t$

(4) $= \lim_{t \to 0} \{f(g_1(z + t), g_2(z + t)] - f[g_1(z), g_2(z + t)] + f[g_1(z), g_2(z + t)] - f[g_1(z), g_2(z)]\}/t$

where we have subtracted and added the same term

(5) $= \lim_{t \to 0} \{f[x_1'', x_2''] - f[x_1', x_2''] + f[x_1', x_2''] - f[x_1', x_2']\}/t$

letting

$x_1' \equiv g_1(z),\ x_1'' \equiv g_1(z + t),\ x_2' \equiv g_2(z),\ x_2'' \equiv g_2(z + t)$

(6) $= \lim_{t \to 0} \{f[x_1'', x_2''] - f[x_1', x_2'']\}/t + \lim_{t \to 0}\{f[x_1', x_2''] - f[x_1', x_2']\}/t$

(7) $= \lim_{t \to 0}\{f_1(x_1^*, x_2'')\,(x_1'' - x_1')\}/t + \lim_{t \to 0}\{f[x_1', x_2''] - f[x_1', x_2']\}/t$

where x_1^* is between $x_1' = g_1(z)$ and $x_1'' = g_1(z + t)$, applying the Mean Value Theorem to $f(x, x_2'')$ regarded as a function of its first variable only and hence the derivative in this case is the first order partial derivative $f_1(x, x_2'')$

(8) $= \lim_{t \to 0}\{f_1(x_1^*, x_2'')\,(x_1'' - x_1')\}/t + \lim_{t \to 0}\{f_2(x_1', x_2^*)\,(x_2'' - x_2')\}/t$

where x_2^* is between $x_2' = g_2(z)$ and $x_2'' = g_2(z + t)$, applying the Mean Value Theorem to $f(x_1', x)$ regarded as a function of its second variable only

(9) $= \lim_{t \to 0} f_1(x_1^*, x_2'') \dfrac{[g_1(z + t) - g_1(z)]}{t} + \lim_{t \to 0} f_2(x_1', x_2^*) \dfrac{[g_2(z + t) - g_2(z)]}{t}$

(10) $= f_1(g_1(z), g_2(z))\, g_1'(z) + f_2(g_1(z), g_2(z))\, g_2'(z)$

since as $t \to 0$, x_1'' and x_1^* tend to $x_1' = g_1(z)$ and x_2'' and x_2^* tend to $x_2' = g_2(z)$.

Q.E.D.

Example 1:

$f(x_1, x_2) \equiv x_1x_2$; $g_1(z) = z$; $g_2(z) = z^2 + 1$

(11) $h(z) \equiv f[g_1(z), g_2(z)] = g_1(z)\, g_2(z)$

$= z(z^2 + 1)$

(12) $= z^3 + z$

(13) Therefore $h'(z) = 3z^2 + 1$

where we have computed the derivative using the direct expression (12). Now we compute the derivative using the composite function chain rule (2) or (10):

$$
\begin{aligned}
h'(z) &= f_1(x_1, x_2)\, g_1'(z) + f_2(x_1, x_2)\, g_2'(z) \\
&= x_2 1 + x_1(2z) \\
&= z^2 + 1 + z\,(2z) \\
&= 3z^2 + 1 \qquad (14)
\end{aligned}
$$

which agrees with (13).

The chain rule (2) seems to be a rather complex way of doing something simple. However, the following example shows how the rule may be used to deduce a formula for the slope of an indifference curve in terms of the first order partial derivatives of a consumer's utility function.

Example 2: $f(x_1, x_2)$ is the consumer's utility function, $x_1 = g_1(z) = z$; $x_2 = g_2(z) = g_2(x_1)$. Set utility equal to a constant; i.e.,

(15) $h(z) \equiv f[g_1(z), g_2(z)]$
$= f[z, g_2(z)] = \text{constant} = u$ say

(16) Therefore $h'(z) = f_1[z, g_2(z)] \bullet 1 + f_2[z, g_2(z)]\, g_2'(z) = 0$

(17) Therefore $g_2'(z) = - f_1[z, g_2(z)] \,/\, f_2[z, g_2(z)]$.

Note that as z varies, $x_1 = z$ and $x_2 = g_2(z)$ is the set of (x_1, x_2) points that give the consumer the constant utility u. For each x_1, the x_2 point that gives this constant level of utility is $x_2 = g_2(x_1)$, and the slope of the indifference curve through this point is

(17) $g_2'(x_1) = - f_1(x_1, x_2) \,/\, f_2(x_1, x_2)$ where $x_2 = g(x_1)$.

As a concrete example of this formula, consider the Cobb-Douglas utility function defined in section 3 with the taste parameter $\alpha = 1/2$. Thus

(18) $u = f(x_1, x_2) \equiv x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$

Using formula (17), the slope of the indifference curve through $x_1 > 0$ and $x_2 > 0$ is

(19) $-f_1(x_1, x_2) \,/\, f_2(x_1, x_2) = -\frac{1}{2}\, x_1^{-\frac{1}{2}} x_2^{\frac{1}{2}} \,/\, \frac{1}{2}\, x_1^{\frac{1}{2}} x_2^{-\frac{1}{2}} = -x_2/x_1$

Note: In order for formula (17) to be valid, we require that $f_2(x_1, x_2) \neq 0$.