7. Single Variable Comparative Statics Analysis and the Envelop Theorem

Recall the unconstrained maximization problem, $\max_x f(x)$, that we studied in section 2. The function f is the economic agent's *objective function* that he or she is trying to maximize and x is the agent's *decision* or *choice* variable. We shall now complicate matters by assuming that the objective function f also depends on a variable "a" that cannot be controlled by the economic agent. Thus the objective function that is to be maximized with respect to x is now f(x, a).

For an example of such a function, consider the f defined by (5) in section 3: there, "a" could be p_1, p_2, I or □. In general, economists are interested in knowing how the optimal x responds to a change in the signal, "a".

Our maximization problem may now be written as:

(1) $\max_x f(x, a)$.

Assuming that f is differentiable, the first order necessary condition for solving (1) is:

(2) $f_1(x, a) = 0$.

We suppose that for an initial "a", the solution to (1) is $x^* = g(a)$ and so we have

(3) $f_1[g(a), a] = 0$.

We also assume that the second order sufficient condition for solving (1) is satisfied at x^*:

(4) $f_{11}[g(a), a] < 0$.

Equation (2) may be solved for x = g(a) and if we knew f precisely, we could determine this solution function g. However, in many cases, we may not know f very accurately, but we may know the signs of the partial derivatives of f up to say the second order (again, recall example 1 in section 3). Under these conditions, we can determine how x = g(a) changes as a changes; more specifically, we can determine the slope of the response function, $dx/da = g'(a)$, as follows. Simply differentiate both sides of equation (3) with respect to a, using the composite function chain rule developed in the previous section. We obtain the following equation:

(5) $f_{11}[g(a), a]\, g'(a) + f_{12}[g(a), a]1 = 0$

(6) or $g'(a) = -f_{12}[g(a),a] \;/\; f_{11}[g(a), a]$.

Thus the response of x to a small change in a hinges on the sign of the second order (cross) partial derivative of the objective function, $f_{12}[g(a), a] = f_{12}(x^*, a)$: if this derivative is positive, then the optimal x^* will increase as a increases; if f_{12} is negative , then the optimal x^* will decrease as a increases. To deduce this rule, we used the second order condition (4).

This is a rather amazing qualitative result and we will see several concrete applications of it later in the course. This qualitative result is an example of *comparative statics analysis.*

There is one additional result that we wish to derive in this section. Define the *optimized objective function* as a function of the signal or stimulus variable "a" by

(7) $h(a) \equiv \max_x f(x, a)$

(8) $\qquad = f[g(a), a]$

Now differentiate (8) with respect to "a" using our multivariate chain rule:

(9) $h'(a) = f_1[g(a), a]g'(a) + f_2[g(a), a]$
$\qquad = 0\ g'(a) + f_2[g(a), a]$ using (3)
$\qquad = f_2[g(a), a].$

Thus to determine the slope of the optimized objective function with respect to "a", we need only partially differentiate $f(x^*, a)$ with respect to its second variable "a". The result (9) is known as the *envelop theorem*.

The results in this section were first derived by the famous American economist, Paul Samuelson, in his book, *Foundations of Economic Analysis*, 1947.

Note that the only difficult mathematical result that was required to derive all of this was the Theorem in section 6.

8. The Geometry of Single Variable Comparative Statics Analysis

Consider the following initial unconstrained maximization problem where the value of the parameter a is a_1:

(1) $\max_x\{f(x, a_1): -\infty < x < +\infty\}.$

Suppose $x_1 = g(a_1)$ solves (1) and we have

(2) $f_1(x_1, a_1) = 0;$

(3) $f_{11}(x_1, a_1) < 0.$

Now consider problem (1) when a_1 is increased by Δa to $a_2 = a_1 + \Delta a$ where $a_2 > a_1$. We suppose $x_2 = g(a_2)$ solves the new problem and we have:

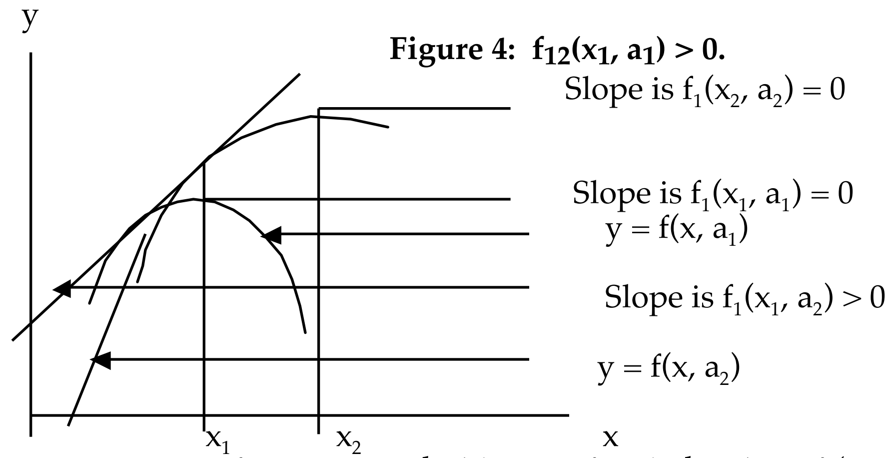
(4) $f_1(x_2, a_2) = 0$;

(5) $f_{11}(x_2, a_2) < 0$

where (5) will follow from (3) and the continuity of f_{11} if Δa is sufficiently small.

We can illustrate graphically the unconstrained maximization problem (1) by plotting $y = f(x, a_1)$ as a function of x, holding a_1 fixed. The geometry of this maximization problem is illustrated in Figure 4 below; see the lower of the two graphs. We can also illustrate the second maximization problem when a_1 is replaced by a_2 by plotting $y = f(x, a_2)$ as a function of x; see the higher graph in Figure 1 below.

Figure 4: $f_{12}(x_1, a_1) > 0$.



As we increase a from a_1 to a_2 holding x_1 fixed, the slope $f_1(x_1, a)$ increases from 0 to the positive number $f_1 = (x_1, a_2)$. Thus $f_{12}(x_1, a_1)$ will be positive and $x_2 = g(a_2)$ will be greater than $x_1 = g(a_1)$; i.e., $g'(a_1) = dx^*(a_1)/da > 0$.

Figure 5 below illustrates the case where $f_{12}(x_1, a_1) = 0$ and $x_1 = x_2$ and $g'(a_1) = 0$ while Figure 6 illustrates the case where $f_{12}(x_1, a_1) < 0$, $x_2 < x_1$ and $g'(a_1) < 0$.

Figure 5: $f_{12}(x_1, a_1) = 0$

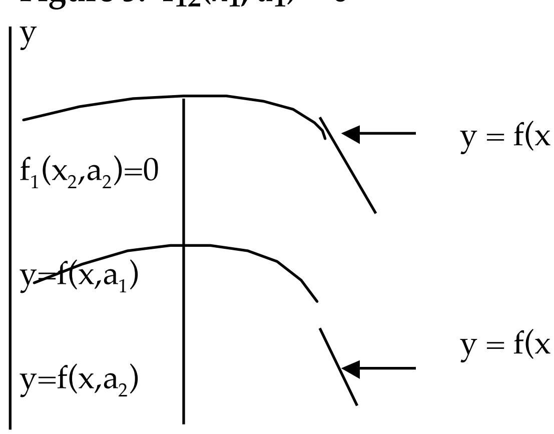
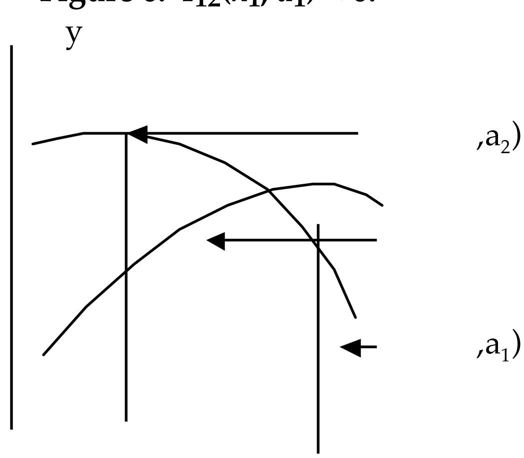


Figure 6: $f_{12}(x_1, a_1) < 0$.



$x_1 = x_2$ x_2 x_1

Thus the sign of the derivative $f_{12}(x_1, a_1)$ tells ~~us whether the optimal x~~ increases or decreases as a increases.

Application to Producer Supply and Demand Functions

Suppose that the maximum output y that a producer can produce in a given time period using the positive amount of input x is

(6) $y = \alpha f(x)$

where $\alpha > 0$ is an efficiency parameter and f is a twice continuously differentiable production function. Given a positive price for a unit of output, $p > 0$, and a positive input price, $w > 0$, we assume that the producer solves the following profit maximization problem:

(7) $\max_{y,\, x}\{py - wx:\ y = \alpha f(x)\} \equiv \pi(p, w, \alpha).$

We can use the constraint to eliminate y from the objective function and we obtain the following unconstrained maximization problem that is equivalent to (7):

(8) $\max_x\{p\alpha f(x) - wx\} \equiv \pi(p, w, \alpha).$

We assume that the solution x^* to (8) satisfies:

(9) $p\alpha f'(x^*) - w = 0;$

(10) $p\alpha f''(x^*) < 0.$

Now regard x^* as a function of p, w and α, say $x^* = d(p, w, \alpha)$. To determine how the demand for input changes as the output price p increases, replace x^* in (9) by $d(p, w, \alpha)$ and differentiate the resulting equation with respect to p. Doing this differentiation using normal calculus rules (treating w and α as constants), we obtain the following equation:

(11) $\alpha f'[d(p, w, \alpha)] + p\alpha f''[d(p, w, \alpha)]\, \dfrac{\partial d}{\partial p}(p, w, \alpha) = 0$ or

(12)
$$\begin{aligned}\frac{\partial d}{\partial p}(p, w, \alpha) &= -\alpha f'(x^*)/p\alpha f''(x^*)\\ &= -f'(x^*)/pf''(x^*)\\ &= (-)(+)/(+)(-) \qquad \text{using (9) and (10)}\\ &> 0.\end{aligned}$$

Thus input demand increases as the output price increases.

To obtain the response of optimal output y* to an increase in the output price, we use the constraint in (7) to define optimal output y* in terms of the optimal input x*; i.e., define the optimal supply function $s(p, w, \alpha)$ as

(13) $s(p, w, \alpha) \equiv \alpha f[d(p, w, \alpha)]$.

Now partially differentiate (13) with respect to p and use (12) to determine the derivative $\partial d(p, w, a)/\partial p$:

$$
\begin{aligned}
(14)\qquad \frac{\partial s}{\partial p}(p, w, \alpha) &= \alpha f'[d(p, w, \alpha)]\,\frac{\partial d}{\partial p}(p, w, \alpha)\\
&= \alpha f'(x^*)\,(-f'(x^*)/p f''(x^*))\\
&= -\alpha\,[f'(x^*)]^2/p f''(x^*)\\
&= (-)\,(+)\,(+)/(+)(-) \qquad \text{using (9) and (10)}\\
&> 0.
\end{aligned}
$$

Thus output supply increases as the output price increases.

To determine how the optimized objective function $\pi(p, w, \alpha)$ changes as p increases, we need only differentiate π defined by (15) with respect to p:

(15) $\pi(p, w, \alpha) \equiv p\alpha f[d(p, w, \alpha)] - w d(p, w, \alpha)$.

Partially differentiating (15) with respect to p yields:

$$\frac{\partial \pi}{\partial p}(p, w, \alpha) = \alpha f(x^*) + [p\alpha f'(x^*) - w]\,\frac{\partial d}{\partial p}(p, w, \alpha)$$

(16) $\qquad = \alpha f(x^*) = s(p, w, \alpha) \qquad$ using (9).

We could have obtained result (16) by using the *Envelop Theorem*: replace f(x, a) by

(17) $F(x, p) \equiv p\alpha f(x) - wx$;

replace $x^* = g(a)$ by $x^* = d(p)$; replace a by p; and replace h(a) by $\pi(p)$ where

(18) $\pi(p) \equiv \max_x\{F(x, p)\}$.

The Envelop Theorem tells us that

(19) $\pi'(p) = F_2(x^*, p) = \alpha f(x^*)$

which is (16).

Problems

1. Adapt the above methodology to obtain formulae for the derivatives $\partial d(p, w, \square)/\partial w$, $\partial d(p, w, \square)/\partial \square$, $\partial s(p, w, \square)/\partial w$ and $\partial s(p, w, \square)/\partial \square$. Sign these derivatives.

2. Show that $\partial \square(p, w, \square)/\partial w = -d(p, w, \square)$ (Hotelling's Lemma).

3. Show that $\partial s(p, w, \square)/\partial w = -\partial d(p, w, \square)/\partial p$ (Hotelling Symmetry Condition).

4. Calculate the consumer's system of demand functions $D_1(p_1, p_2, I)$ and $D_2(p_1, p_2, I)$ if the consumer's utility function is defined by:

(i) $u(x_1, x_2) \equiv f\ [x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}]$

where f is a continuously differentiable function of one variable which has $f'(x) > 0$ for all $x > 0$.

5. Solve $\max_x \{f(x){:}\ x \geq 0\}$ for the following functions f:

(a) $f(x) \equiv -x^2 + 2x - 2$
(b) $f(x) \equiv \ln x - x + 1$
(c) $f(x) \equiv -x^2 - 2x$.

Check the relevant second order conditions.

6. Solve $\max_{x_1, x_2}\{(f(x_1, x_2)\}$ for the following f:

(a) $f(x_1, x_2) \equiv -\ x_1^2 + x_1 x_2 - x_2^2 + 2$;
(b) $f(x_1, x_2) \equiv \square \ln x_1 + \ln x_2 - 2x_1 - 2x_2 + 2$.

Check the relevant second order conditions.

9. The Directional Derivative and First Order Necessary Conditions.

Let $f(x_1, x_2, \ldots, x_N)$ be a function of N variables. In order to define the directional derivative of f, we first need to define a direction.

Definition 1: A direction v is defined to be N numbers $v_1, v_2, \ldots, v_N$ whose squared components sum to 1; i.e., $v_1^2 + v_2^2 + \ldots + v_N^2 = 1$. Thus a direction $v \equiv (v_1, v_2, \ldots, v_N)$ in N dimensional space is a point on the sphere of radius 1 with center at the origin.

Definition 2: The direction derivative of the function f evaluated at the point $(x_1, x_2, \ldots, x_N)$ in the direction v, denoted as $D_v f(x_1, x_2, \ldots, x_N)$, is defined as the following limit (if it exists):

$$(1) \qquad D_v f(x_1, x_2, \ldots, x_N) \equiv \lim_{t \to 0} \frac{f(x_1 + tv_1, \ldots, x_N + tv_N) - f(x_1, \ldots, x_N)}{t}$$

Geometric Interpretation?

Let $e_i \equiv (0, \ldots, 0, 1, 0, \ldots, 0)$ denote the point in N dimensional space which has ith coordinate equal to 1 and all other coordinates equal to zero. (This is often called the ith unit vector in matrix algebra).

Suppose we chose our direction v to be e_i. Then using definition 1, it can be seen that

$$(2) \qquad D_{e_i} f(x_1, x_2, \ldots, x_N) = f_i(x_1, x_2, \ldots, x_N)$$

where f_i denotes the ith first order partial derivative of f. Thus partial derivatives are special cases of the directional derivative: the ith partial derivative is equal to the directional derivative in the direction given by the ith coordinate axis.

Example: Let $f(x_1, x_2) = x_1^2 + x_2$ and $v = (1/\sqrt{2}, 1/\sqrt{2}) = (2^{-\frac{1}{2}}, 2^{-\frac{1}{2}})$

$$\begin{aligned}
D_{(2^{-\frac{1}{2}}, 2^{-\frac{1}{2}})} f(x_1, x_2) &\equiv \lim_{t \to 0} [f(x_1 + t2^{-\frac{1}{2}}, x_2 + t2^{-\frac{1}{2}}) - f(x_1, x_2)] / t \\
&= \lim_{t \to 0} [(x_1 + 2^{-\frac{1}{2}} t)^2 + (x_2 + 2^{-\frac{1}{2}} t) - (x_1^2 + x_2)] / t \\
&= \lim_{t \to 0} [x_1^2 + 2^{1-\frac{1}{2}} t x_1 + 2^{-1} t^2) + (x_2 + 2^{-\frac{1}{2}} t) - x_1^2 - x_2] / t \\
&= \lim_{t \to 0} [2^{\frac{1}{2}} t x_1 + 2^{-1} t^2 + 2^{-\frac{1}{2}} t] / t \\
&= \lim_{t \to 0} [2^{\frac{1}{2}} x_1 + 2^{-1} t + 2^{-\frac{1}{2}}]
\end{aligned}$$

$$(3) \qquad = 2^{\frac{1}{2}} x_1 + 2^{-\frac{1}{2}}$$

The above example shows that it is not very easy to calculate a directional derivative in general. This contrasts to the case of partial derivatives where ordinary calculus rules for differentiation could be used. Thus we need an easier way for calculating directional derivatives, and the following Theorem does this for us.

First Order Directional Derivative Theorem. If the first order partial derivatives of f exist and are continuous functions in a neighbourhood around the point $x_1, x_2, \ldots, x_N$, then the directional derivative of f evaluated at $x_1, \ldots, x_N$ for any direction $v \equiv (v_1, v_2, \ldots, v_N)$ exists and is given by the following weighted sum of partial derivatives:

$$(4) \qquad D_v f(x_1, x_2, \ldots, x_N) = \sum_{i=1}^{N} v_i f_i(x_1, x_2, \ldots, x_N)$$
$$= v_1 f_1(x_1, \ldots, x_N) + v_2 f_2(x_1, \ldots, x_N) + \ldots + v_N f_N(x_1, \ldots, x_N).$$

Proof: For simplicity, we prove the result for the case where N = 2. The general case may be proven in a similar manner.

By the definitional of the directional derivative, we have

$$D_v f(x_1, x_2) \equiv \lim_{t \to 0} [f(x_1 + tv_1, x_2 + tv_2) - f(x_1, x_2)] \;/\; t$$

$$= \lim_{t \to 0} [f(x_1 + tv_1, x_2 + tv_2) - f(x_1, x_2 + tv_2) + f(x_1, x_2 + tv_2) - f(x_1, x_2)] \;/\; t$$

upon adding and subtracting the term $f(x_1, x_2 + tv_2)$

$$= \lim_{t \to 0} [f_1(x_1^*, x_2 + tv_2)\,(x_1 + tv_1 - x_1) + f_2(x_1, x_2^*)(x_2 + tv_2 - x_2)] \;/\; t$$

applying the Mean Value Theorem twice where x_1^* is between $x_1 + tv_1$ and x_1 and x_2^* is between $x_2 + tv_2$ and x_2

$$= \lim_{t \to 0} [f_1(x_1^*, x_2 + tv_2)v_1 + f_2(x_1, x_2^*)v_2$$

canceling terms involving t

$$= f_1(x_1, x_2)v_1 + f_2(x_1, x_2)v_2$$

taking limits and using the continuity of f_1 and f_2

$$= v_1 f_1(x_1, x_2) + v_2 f_2(x_1, x_2).$$

Q.E.D.

Example: Let $f(x_1, x_2) = x_1^2 + x_2$ and $v_1 = 2^{-\frac{1}{2}}$, $v_2 = 2^{-\frac{1}{2}}$.

Then

$$D_v f(x_1, x_2) = v_1 f_1(x_1, x_2) + v_2 f_2(x_1, x_2)$$
$$= 2^{-\frac{1}{2}}\, 2x_1 + 2^{-\frac{1}{2}} \bullet 1$$
$$= 2^{\frac{1}{2}} x_1 + 2^{-\frac{1}{2}}$$

which agrees with (3).

Now that we have introduced the concept of the directional derivative, the following condition for an interior max or min should be obvious.

First Order Necessary Conditions **for f to attain a local interior max or min** at $x_1, x_2, \ldots, x_N$ are:

(5) $D_v f(x_1, x_2, \ldots, x_N) = 0$ for every direction $v = (v_1, v_2, \ldots, v_N)$.

If the regularity conditions for the First Order Directional Derivative Theorem are satisfied, then by (4), conditions (5) are equivalent to:

(6) $\sum_{i=1}^{N} v_i f_i(x_1, x_2, \ldots, x_N) = 0$ for every $v = (v_1, v_2, \ldots, v_N)$ such that $\sum_{i=1}^{N} v_i^2 = 1$.

At first sight, conditions (5) and (6) may not seem to be of much practical value, since as soon as $N \geq 2$, we must check (5) and (6) for an **infinite** number of directions v. However, it is easy to verify that conditions (6) are equivalent to our old first order necessary conditions, (3) in section 4, which we now rewrite as conditions (7):

(7)
$$\begin{aligned} \partial f(x_1, x_2, \ldots, x_N)/\partial x_1 &\equiv f_1(x_1, x_2, \ldots, x_N) = 0 \\ \vdots \qquad\qquad & \qquad\qquad \vdots \qquad\qquad \vdots \\ \partial f(x_1, x_2, \ldots, x_N)/\partial x_N &\equiv f_N(x_1, x_2, \ldots, x_N) = 0 \end{aligned}$$

It is obvious that (7) implies (6). To show that (6) implies (7), choose v to be the unit vector e_i for $i = 1, 2, \ldots, N$.

The reader may well be a bit confused at this point. Why did we bother to introduce the concept of the directional derivative if in the end, we simply end up with conditions (7), which we derived before using only the much simpler concept of a partial derivative?

There are two answers to this questions: (i) if the partial derivative functions are continuous, we now know that conditions (7) imply the seemingly much stronger conditions (5), and (ii) in order to derive valid second order sufficient conditions, we need the concept of the directional derivative.