CHAPTER 2: ELEMENTARY MATRIX ALGEBRA

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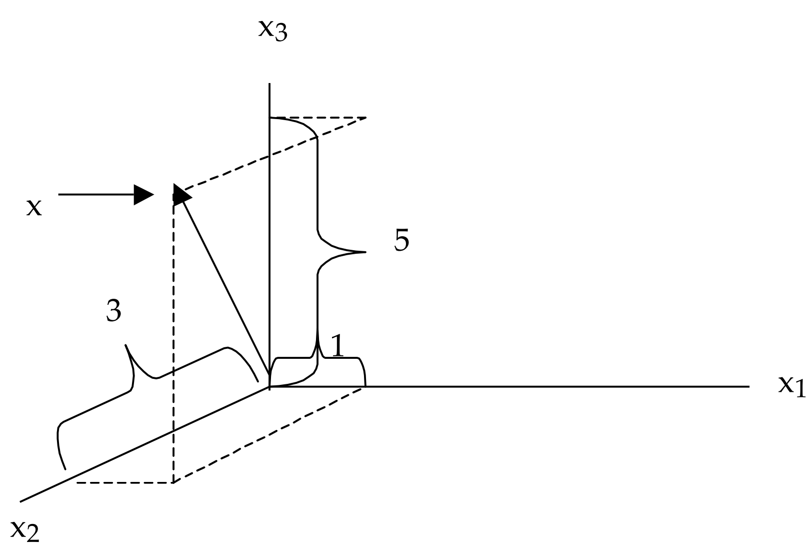
Reference: G. Hadley: *Linear Algebra*, Addison-Wesley, 1961.

1. The Algebra of Vectors and Matrices

Definition: An N *vector* is a column of N numbers, e.g. $x = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ is a 3 dimensional vector.

Geometrically speaking, with each vector x, we can associate a point in N dimensional space; the ith component of the vector x corresponds to the distance along the ith coordinate axis from the point x to the origin $0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

e.g. $x = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$



Definition: An M x N *matrix* A is a rectangular array of numbers with M rows and N columns. e.g.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & & \vdots \\ a_{M1} & a_{M2} & & a_{MN} \end{bmatrix}$$

where a_{ij} is a number for i = 1, 2, . . ., M (row index) and j = 1, 2, . . ., N (column index)

Note that a vector can now be defined to be a matrix with only 1 column.

There are various operations which we can perform on matrices (and vectors):

Definition of Matrix Addition: If A and B are two M by N matrices, then

$$A + B \equiv \begin{bmatrix} a_{11} + b_{11}, & a_{12} + b_{12} & , \ldots, & a_{1N} + b_{1N} \\ \vdots & & & \\ a_{M1} + b_{M1}, & a_{M2} + b_{M2}, & \ldots, & a_{MN} + b_{MN} \end{bmatrix}$$

i.e., we simply add the corresponding components of the two matrices.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 5 & 6 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

then

$$A + B = \begin{bmatrix} 1 & 3 & 4 \\ 6 & 6 & 9 \end{bmatrix}$$

e.g. $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, A + B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Definition of Scalar Multiplication: If A is an M by N matrix and α is a scalar (i.e., a number), then

$$\alpha A \equiv \begin{bmatrix} \alpha a_{11}, \alpha a_{12}, \ldots, \alpha a_{1N} \\ \vdots \\ \alpha a_{M1}, \alpha a_{M2}, \ldots, \alpha a_{MN} \end{bmatrix}$$

e.g. $\alpha = 2,\ A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \alpha A = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Definition of Matrix Multiplication: If A is an M by N matrix and B is an N by K matrix (note that the number of columns in A is equal to the number of rows in B), then

$$AB = \begin{bmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & & \\ a_{M1} & \cdots & a_{MN} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1K} \\ \vdots & & \vdots \\ b_{N1} & \cdots & b_{MK} \end{bmatrix}$$

$$\equiv \begin{bmatrix} c_{11} & \cdots & c_{1K} \\ \vdots & & \vdots \\ c_{M1} & & c_{MK} \end{bmatrix} \text{ where } c_{mk} \equiv \sum_{n=1}^{N} a_{mn} b_{n_k}$$

Thus we end up with M by K matrix.

e.g. $A = \begin{bmatrix} 1 & 2 & 4 \\ 5 & 6 & 8 \end{bmatrix}, B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, AB = \begin{bmatrix} 1x_1 + 2x_2 + 4x_3 \\ 5x_1 + 6x_2 + 8x_3 \end{bmatrix}$

We can now begin to see why matrix algebra is useful in the study of systems of simultaneous linear equations. For example suppose we were given the following system of M equations in the N unknowns $x_1, x_2, \ldots, x_N$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N &= b_2 \\ &\vdots \\ a_{M1}x_1 + a_{M2}x_2 + \ldots + a_{MN}x_N &= b_M \end{aligned}$$

(The a_{ij}'s and the b_M's are given fixed numbers).

The above system of equations can be written much more compactly using matrix notation as

$$Ax = b \qquad \text{where} \qquad A = \begin{bmatrix} a_{11} \cdots a_{1N} \\ \vdots \\ a_{M1} \cdots a_{MN} \end{bmatrix}, x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, b = \begin{bmatrix} b_1 \\ \vdots \\ b_M \end{bmatrix}$$

We are implicitly making use of another definition in the above representation $Ax = b$:

Definition of Matrix Equality: Two M by N matrices A, B are equal (written A = B), if and only if the corresponding components of A and B are equal, i.e., if we have

$$a_{ij} = b_{ij} \qquad \text{for} \qquad \begin{matrix} i = 1, \ldots, M \\ j = 1, \ldots, N. \end{matrix}$$

There are a few more definitions which will be useful in what follows.

Definition: An M by N matrix A is *square* is M = N: i.e., if the number of rows = number of columns.

(Typically in the system of simultaneous linear equations Ax = b, we have A square; i.e., the number of equations = the number of unknowns).

Definition: The N by N *identity* matrix I_N (or I for short) is the following matrix:

$$I_N \equiv \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \dots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

N rows with zeros everywhere except on the main diagonal

Definition of the Transpose of a Matrix: Let $A = \begin{bmatrix} a_{11} a_{12} \dots a_{1N} \\ \vdots \\ a_{M1} a_{M2} \dots a_{MN} \end{bmatrix}$. Then

$$A^T = \begin{bmatrix} a_{11} & \dots & a_{M1} \\ a_{12} & \dots & a_{M1} \\ \vdots & & \vdots \\ a_{1N} & \dots & a_{MN} \end{bmatrix}$$

i.e., the rows and columns of A have been interchanged.

In order to acquire some facility in working with matrices, students are required to do the following problems.

Problem 1: Let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

(i) Calculate AB + 5I - 1C

(ii) Show IA = AI = A

(iii) Show that $(AB)^T = B^T A^T$

(iv) Does AB = BA?

(v) (AB)C = A(BC)?

Problem 2: (More difficult). Let $A = \begin{bmatrix} a_{11} & \dots & a_{1N} \\ \vdots & & \\ a_{M1} & \dots & a_{MN} \end{bmatrix}$ be an M by N matrix

$B = \begin{bmatrix} b_{11} & \dots & b_{1K} \\ \vdots & & \vdots \\ b_{N1} & \dots & b_{NK} \end{bmatrix}$ be an N by K matrix, and $C = \begin{bmatrix} c_{11} & \dots & c_{IL} \\ \vdots & & \vdots \\ c_{K1} & \dots & c_{KL} \end{bmatrix}$ be a K by L matrix.

Show that (AB)C = A(BC). *Hint:* Take the ijth element of (AB)C (which is equal to $\left[\sum_{n=1}^{N} a_{in}b_{n1}, \sum_{n=1}^{N} a_{in}b_{n2}, \dots, \sum_{n=1}^{N} a_{in}b_{nK}\right] \begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{Kj} \end{bmatrix}$ and show that it is equal to the ijth element of A(BC).

Definition: Let A be a square N by N matrix. Then we say that an N by N B matrix is a (left) inverse of A if

$BA = I_N$

An N by N matrix C is a (right) inverse of A if

$AC = I_N$.

There are two points to note about the last definition:

(i) the definition says nothing about whether a left or right inverse for A will in fact exist (they don't always as we shall see) and

(ii) we are forced at this stage to consider both left and right inverse as separate entities (if they exist) since problem 1 (iv) shows that it is not always the case that AB = BA.

Now if we consider our system of simultaneous linear equations Ax = b (where A is square), it is clear that if we knew what a left inverse for A was (call it B assuming one exists), then if we premultiply both sides of the matrix equation Ax = b by B, we obtain:

$BAx = Bb$

or $Ix = Bb$

or $x = Bb$ (recall Problem 1 (ii))

and thus we have a *solution* to the system of equations $Ax = b$, namely $x^* = Bb$ where B is a left inverse for A.

Thus we are interested in two questions:

(i) when will a left inverse for a square matrix A exist?

(ii) how can we compute it if it exists?

It turns out that the notion of a *determinant* is useful in answering both questions.

2. Determinants and their Properties

We must first make some preliminary definitions before we define the determinant of an arbitrary *square* N by N matrix A.

Definition: A *transposition* of the integers $(i_1, i_2, \ldots, i_N)$ is a simple interchange of 2 of the numbers. e.g. (3, 2, 1) is a transposition of (3, 1, 2)

Definition: A permutation of the integers $(1, 2, \ldots, N)$, say $(i_1, i_2, \ldots, i_N)$ is an *even* permutation if it is obtained from $(1, 2, \ldots, N)$ by an even number of transpositions . . . is an *odd* permutation if it can be obtained from $(1, 2, \ldots, N)$ by an odd number of transpositions. e.g. (4, 1, 3, 2) is it an even or odd permutation? What we do is we start with (1, 2, 3, 4) and build towards (4, 1, 3, 2) by making a sequence of transpositions, filling in the appropriate elements starting at the left and working towards the right.

(1, 2, 3, 4)
(4, 2, 3, 1) First transposition, first component is 4 now
(4, 1, 3, 2) Second transposition, second component is 1 now.

No further transpositions are required so as it took only 2 transpositions from (1, 2, 3, 4) to attain (4, 1, 3, 2), thus the permutation is *even*.

Definition: $\epsilon(i_1, i_2, \ldots, i_N) \equiv \begin{cases} 1 \text{ if the permutation } (i_1, i_2, \ldots, i_N) \text{ is even} \\ -1 \text{ if the permutation } (i_1, i_2, \ldots, i_N) \text{ is odd} \end{cases}$
where $(i_1, i_2, \ldots, i_N)$ is some permutation of the integers $(1, 2, \ldots, N)$.

Definition: The *determinant* of an (N by N) *square* matrix A is defined as:

$$\text{Det } A \text{ or } |A| \equiv \sum_{\substack{\text{over all} \\ \text{permutations} \\ (i_1, i_2, \ldots, i_N)}} \epsilon(i_1, i_2, \ldots, i_N)\, a_{1i_1} a_{2i_2} \ldots a_{Ni_N}$$

E.g. $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\quad |A| = \epsilon(1, 2)\, a_{11}a_{22} + \epsilon(2, 1)a_{12}a_{21}$

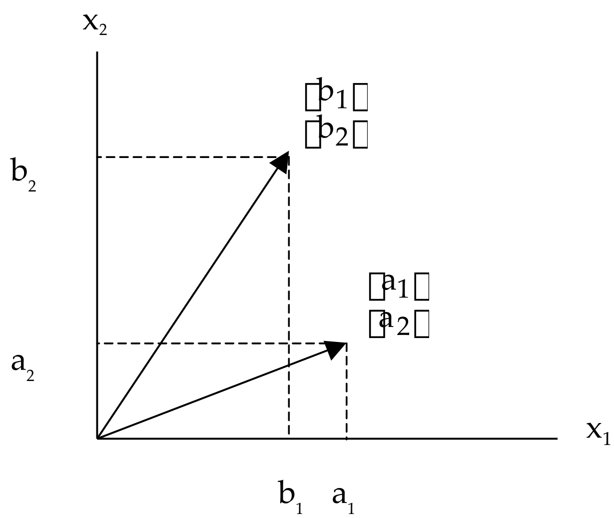
$$= \quad + a_{11}a_{22} - a_{12}a_{21}$$

As there are only 2! = 2 permutations of 2 numbers, a 2 by 2 matrix has a determinant with only 2 terms

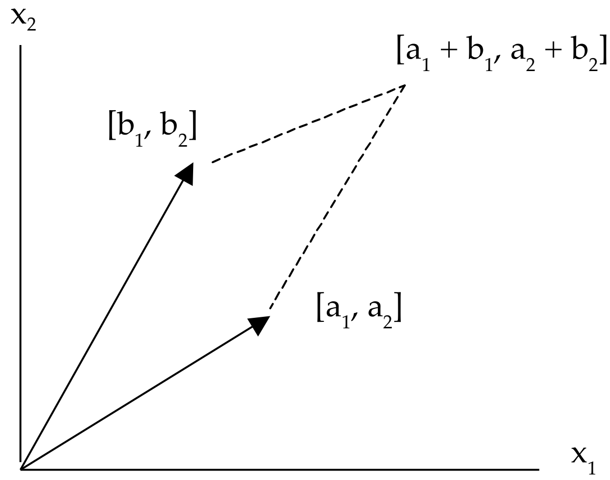
$$\text{E.g. } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; \quad |A| = \begin{array}{ll} \epsilon(1,2,3)a_{11}a_{22}a_{33} & (+) \\ +\ \epsilon(2,1,3)a_{12}a_{21}a_{33} & (-) \\ +\ \epsilon(3,2,1)a_{13}a_{22}a_{31} & (-) \\ +\ \epsilon(1,3,2)a_{11}a_{23}a_{32} & (-) \\ +\ \epsilon(2,3,1)a_{12}a_{23}a_{31} & (+) \\ +\ \epsilon(3,1,2)a_{13}a_{21}a_{32} & (+) \end{array}$$

Thus a determinant is a mapping from an N by N array of numbers into a number.

Problem 3: (Difficult) Let $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$. Let $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ be points in 2 dimensional space; e.g.



Now use the origin $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and the 2 points $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ to form a parallelogram.



Show that the area of the parallelogram is equal to $|A|$ (except possibly for sign).

Hint: the area of a parallelogram is equal to the product of the base times the height.

Comment: the above property of 2 by 2 determinants generalizes to the N by N case. Write the matrix A as N column vector of dimension N, i.e.,

$$A = [A_{\bullet 1}, A_{\bullet 2}, \ldots, A_{\bullet N}] \text{ where } A_{\bullet n} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{Nn} \end{bmatrix} \text{ for } n = 1, \ldots, N.$$

Now join each of the N points $A_{\bullet n}$ to the origin and let these N vectors form the edges of a parallelepiped (the N dimensional generalization of a parallelogram). Then the volume which this parallelepiped encloses = $|A|$. We will prove this fact later after we have developed the concept of orthogonality (i.e., perpendicularity).

Some Useful Properties of Determinants

Lemma 1: If two rows of the N by N matrix A are identical, then $|A| = 0$. ($N \geq 2$).

Proof: Look at $|A| = \sum_{\substack{\text{permutations} \\ (i_1, i_2, \ldots, i_N)}} \varepsilon(i_1, i_2, \ldots, i_N)\, a_{1i_1} a_{2i_2} \cdots a_{Ni_N}$

Let us suppose that rows 1 and 2 of A are identical. Pick an arbitrary term in the above summation, say $\epsilon(i_1, i_2, \ldots, i_N)\, a_{1i_1} a_{2i_2} \ldots a_{Ni_N}$. Then the term $\epsilon(i_2, i_1, \ldots, i_N)\, a_{1i_2} a_{2i_1} \ldots a_{Ni_N}$ is equal in absolute value to the first term (since rows 1 and 2 are identical and thus $a_{1i_1} a_{2i_2} = a_{2i_1} a_{1i_1} = a_{1i_2} a_{2i_1}$) but it will be of opposite sign to the first term since the function ϵ changes sign every time we make a transposition. We may carry on dividing the above summation into two separate summations, where each term in the first summation has a corresponding term in the second summation of opposite sign and thus $|A| = 0$.

Q.E.D.

Lemma 2: $|A^T| = |A|$ where A is an N by N matrix; i.e., the determinant of the transposed matrix is the same as the determinant of the matrix itself.

Proof:

$$|A| = \sum_{(i_1, i_2, \ldots, i_N)} \epsilon(i_1, i_2, \ldots, i_N)\, a_{1i_1} a_{2i_2} \ldots a_{Ni_N}$$

$$= \sum_{(i_1, i_2, \ldots, i_N)} \epsilon(i_1, i_2, \ldots, i_N)\, a_{j_1 1} a_{j_2 2} \ldots a_{j_N N}$$

$\uparrow$

where we have rearranged terms so that the column subscript appears in natural order. Now if $(i_1, i_2, \ldots, i_N)$ were an even permutation, it follows that $(j_1, j_2, \ldots, j_N)$ is also even permutation since as we rearrange the indices $(i_1, i_2, \ldots, i_N)$ by successive transpositions into $(1, 2, \ldots, N)$, we are simultaneously transposing $(1, 2, \ldots, N)$ into $(j_1, j_2, \ldots, j_N)$. Thus we have

$$= \sum_{(j_1, j_2, \ldots, j_N)} \epsilon(j_1, j_2, \ldots, j_N)\, a_{j_1 1} a_{j_2 2} \ldots a_{j_N N}$$

$$= |A^T|.$$

Q.E.D.

The above two lemmas imply that if two *columns* of the N by N matrix A are identical, then $|A| = 0$.

Lemma 3: Let A by an N by N matrix, which has nth row equal to $A_{n\bullet}$; i.e. $A = \begin{bmatrix} A_{1\bullet} \\ A_{2\bullet} \\ \vdots \\ A_{N\bullet} \end{bmatrix}$. Let k be a scalar. Then $|[kA_{1\bullet}^T, A_{2\bullet}^T, \ldots, A_{N\bullet}^T]| = k|A|$, i.e., if we multiply a row of the matrix A by a scalar k, then the determinant of the resulting matrix $= k|A|$.

Proof: $\left|\begin{bmatrix} kA_{1\bullet} \\ A_{2\bullet} \\ \vdots \\ A_{N\bullet} \end{bmatrix}\right| = \sum_{(i_1, i_2, \ldots, i_N)} \varepsilon(i_1, i_2, \ldots, i_N)\,(ka_{1i_1})\, a_{2i_2} \ldots a_{Ni_N}$

$$= k \sum_{(i_1, \ldots, i_N)} \varepsilon(i_1, \ldots, i_N)\, a_{1i_1} a_{2i_2} \ldots a_{Ni_N}$$

$$= k \; |A|$$

Q.E.D.

Lemma 4: $\left|\begin{bmatrix} A_{1\bullet} + B_{1\bullet} \\ A_{2\bullet} \\ \vdots \\ A_{N\bullet} \end{bmatrix}\right| = \left|\begin{bmatrix} A_{1\bullet} \\ A_{2\bullet} \\ \vdots \\ A_{N\bullet} \end{bmatrix}\right| + \left|\begin{bmatrix} B_{1\bullet} \\ A_{2\bullet} \\ \vdots \\ A_{N\bullet} \end{bmatrix}\right|$ where

$A_{1\bullet} = [a_{11} \ldots a_{1N}]$
$B_{1\bullet} = [b_{11} \ldots b_{1N}]$
$A_{2\bullet} = [a_{21} \ldots a_{2N}]$
$\vdots$
$A_{N\bullet} = [a_{N1} \ldots a_{NN}]$

Proof: $\left|\begin{bmatrix} A_{1\bullet} + B_{1\bullet} \\ A_{2\bullet} \\ \vdots \\ A_{N\bullet} \end{bmatrix}\right| \equiv \sum_{(i_1, i_2, \ldots, i_N)} \varepsilon(i_1, i_2, \ldots, i_N)\,(a_{1i_1} + b_{1i_1})\, a_{2i_2} \ldots a_{Ni_N}$

$$= \sum_{(i_1, i_2, \ldots, i_N)} \varepsilon(i_1, i_2, \ldots, i_N)\, a_{1i_1} a_{2i_2} \ldots a_{Ni_N}$$

$$+ \sum_{(i_1, i_2, \ldots, i_N)} \varepsilon(i_1, i_2, \ldots, i_N)\, b_{1i_1} a_{2i_2} \ldots a_{Ni_N}$$

$$= \left|\begin{bmatrix} A_{1\bullet} \\ A_{2\bullet} \\ \vdots \\ A_{N\bullet} \end{bmatrix}\right| + \left|\begin{bmatrix} B_{1\bullet} \\ A_{2\bullet} \\ \vdots \\ A_{N\bullet} \end{bmatrix}\right|$$

Q.E.D.

Problem 4: Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ Show $|AB| = |A| \bullet |B|$.

Problem 5: Suppose we are given N-1, N dimensional vectors

$$A_{\bullet 2} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{N2} \end{bmatrix}, \; A_{\bullet 3} = \begin{bmatrix} a_{13} \\ a_{23} \\ \vdots \\ a_{N3} \end{bmatrix}, \; \ldots, \; A_{\bullet N} = \begin{bmatrix} a_{1N} \\ a_{2N} \\ \vdots \\ a_{NN} \end{bmatrix}$$

and we define $A_{\bullet 1}$ by $A_{\bullet 1} \equiv k_2 A_{\bullet 2} + k_3 A_{\bullet 3} + \ldots + k_N A_{\bullet N}$ where $k_2, k_3, \ldots, k_N$ are numbers. Show that $|A| = |A_{\bullet 1}, A_{\bullet 2}, \ldots, A_{\bullet N}| = 0$.

Hint: Use the definition of $A_{\bullet 1}$ above and the previous 4 lemmas.

Lemma 5: Let A be an N by N matrix. If two columns of A are interchanged and if we take the determinant of the resulting matrix, then the resulting determinant $= - |A|$.

Proof:

$$0 = | A_{\bullet 1} A_{\bullet 2}, \ldots, A_{\bullet i-1}, A_{\bullet i} + A_{\bullet j}, A_{\bullet i+1}, \ldots, A_{\bullet j-1}, A_{\bullet j} + A_{\bullet i}, A_{\bullet j+1}, \ldots, A_{\bullet N} |$$

(the determinant is 0 since 2 columns of the above matrix are equal; recall lemmas (1) and (2))

$$= | A_{\bullet 1} \ldots A_{\bullet i-1}, A_{\bullet i}, A_{\bullet i+1}, \ldots, A_{\bullet j} + A_{\bullet i}, \ldots, A_{\bullet N} | + | A_{\bullet 1}, \ldots, A_{\bullet i-1}, A_{\bullet j},$$

$$A_{\bullet i+1}, A_{\bullet j} + A_{\bullet i}, \ldots, A_{\bullet N} |$$

(using lemma (4) in conjunction with lemma (2)).

$$= | A_{\bullet 1} \ldots A_{\bullet i} \ldots A_{\bullet j} \ldots A_{\bullet N} | + | A_{\bullet 1} \ldots A_{\bullet i} \ldots A_{\bullet i} \ldots A_{\bullet N} |$$
$$+ | A_{\bullet 1} \ldots A_{\bullet j} \ldots A_{\bullet j} \ldots A_{\bullet N} | + | A_{\bullet 1} \ldots A_{\bullet j} \ldots A_{\bullet i} \ldots A_{\bullet N} |$$

(again using lemma (4) in conjunction with lemma (2) 2 times)

$$= | A_{\bullet 1} \ldots A_{\bullet i} \ldots A_{\bullet j} \ldots A_{\bullet N} | + 0$$
$$+ 0 + | A_{\bullet 1} \ldots A_{\bullet j} \ldots A_{\bullet i} \ldots A_{\bullet N} |$$

Since matrices which have 2 identical columns have 0 determinants

Therefore $| A_{\bullet 1} \ldots A_{\bullet i} \ldots A_{\bullet j} \ldots A_{\bullet N} | = - | A_{\bullet 1} \ldots A_{\bullet j} \ldots A_{\bullet i} \ldots A_{\bullet N} |$

Q.E.D.

Lemma 6: Let A and B be 2 N x N matrices. Then $|AB| = |A| \bullet |B|$.

Proof: Let $A = [A_{\bullet 1}, A_{\bullet 2}, \ldots A_{\bullet N}]$ and $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1N} \\ b_{21} & b_{22} & \cdots & b_{2N} \\ \vdots & & & \\ b_{N1} & b_{N1} & \cdots & b_{NN} \end{bmatrix}$

then

$$|AB| = \left| \sum_{k_1=1}^{N} A_{\bullet k_1} b_{k_1 1}, \sum_{k_2=1}^{N} A_{\bullet k_2} b_{k_2 2}, \ldots, \sum_{k_N=1}^{N} A_{\bullet k_N} b_{k_N N} \right|$$

by definition of AB

$$= \sum_{k_1=1}^{N} \sum_{k_2=1}^{N} \cdots \sum_{k_N=1}^{N} |A_{\bullet k_1} b_{k_1 1}, A_{\bullet k_2} b_{k_2 2}, \ldots, A_{\bullet k_N} b_{k_N N}|$$

making repeated use of lemma (4)

$$= \sum_{k_1=1}^{N} \sum_{k_2=1}^{N} \cdots \sum_{k_N=1}^{N} b_{k_1 1}, b_{k_2 2}, \ldots, b_{k_N N} |A_{\bullet k_1}, A_{\bullet k_2}, \ldots, A_{\bullet k_N}|$$

making repeated used of lemma (3) (applied to transposes)

$$= \sum b_{k_1 1} b_{k_2 2} \ldots b_{k_N N} |A_{\bullet k_1}, A_{\bullet k_2}, \ldots, A_{\bullet k_N}|$$

over all permutations
$(k_1, k_2, \ldots, k_N)$
of $(1, 2, \ldots, N)$

since by lemma (1), the determinant is zero if any two rows (or columns using lemma (2)) are identical.

$$= \sum b_{k_1 1} b_{k_2 2} \ldots b_{k_N N} \epsilon(k_1, k_2, \ldots, k_N) |A_{\bullet 1}, A_{\bullet 2}, \ldots, A_{\bullet N}|$$

permutations
$(k_1, k_2, \ldots, k_N)$

since by lemma (5) every time we interchange a column of A, we change the sign of the determinant . . .

$= |B^T| \bullet |A_{\bullet 1} A_{\bullet 2} \ldots A_{\bullet N}| = |B| \bullet |A|$ (using the definition of) $|B^T|$

$= |A| \bullet |B|$

since |B| and |A| are scalars the order of multiplication can be interchanged.

Q.E.D.

We note that the calculation of the determinant of a square matrix A using the permutation definition on page 10 above is not an easy matter if the size of A is greater than say 5 since when N = 5, the number of terms in the definition equals 120 = $1 \times 2 \times 3 \times 4 \times 5$. When N = 10, the number of terms is 10! = 4, 536,000. Hence, in the next section, we develop a practical method for calculating determinants. The method relies on the properties of determinants that we developed in this section.

3. A Gaussian Method for Calculating Determinants

A *diagonal method* $A \equiv [a_{ij}]$ is a square matrix that has zero elements everywhere except possibly down the main diagonal which runs from the northwest corner of the matrix to the southeast corner; i.e., $A \equiv [a_{ij}]$ is diagonal iff $a_{ij} = 0$ for all $i \neq j$.

From the definition of the determinant, it is clear that the determinant of a diagonal matrix is equal to the product of its main diagonal elements; i.e.,

$$|A| = \prod_{i=1}^{N} a_{ii} \qquad \text{if A is diagonal.}$$

An *upper triangular matrix* $A \equiv [a_{ij}]$ is a square matrix that has zero elements *below* the main diagonal; i.e., $a_{ij} = 0$ for all i and j such that $1 \leq j < i \leq N$.

It can be seen that the determinant of an upper triangular matrix is also equal to the product of its main diagonal elements, $\prod_{i=1}^{N} a_{ii}$. Why is this? Recall the definition of |A|:

$$|A| = \sum_{\text{permutations}} (j_1, j_2, \ldots, j_N) a_{1j_1} a_{2j_2} \cdots a_{Nj_N}.$$

Assume A is upper triangular. If we pick $j_1 \geq 2$, then eventually, one of the later indexes $j_2, j_3, \ldots, j_N$ must be chosen to be 1. For the sake of definiteness, suppose $j_2 = 1$ and hence a_{21} appears in the term under consideration. But since all elements below a_{11} in the first column of the matrix are 0, this term must be 0. Thus, in order to obtain a nonzero term, we must pick $j_1 = 1$. Now consider the choices for the j_2 index. Since we have chosen $j_1 = 1$ in order to obtain nonzero terms, j_2 can be any one of the indexes 2, 3, . . ., N. However, if we pick $j_2 \geq 3$, then one of the later indexes $j_3, j_4, \ldots, j_N$ must be chosen to be 2 in order for $(1, j_2, j_3, \ldots, j_N)$ to be a permutation of (1, 2, . . ., N). For the sake of definiteness, suppose $j_3 = 2$ and hence a_{32} appears in the term under consideration. But since all elements below a_{22} in the second column must be zero by the definition of an upper triangular matrix, we have $a_{32} = 0$. Thus in order to obtain a nonzero term in the definition of the determinant, we must pick $j_1 = 1$ and $j_2 = 2$. The same logic can be repeated to show that the only possible nonzero term in the definition of the determinant of an upper triangular A is the term where $(j_1, j_2, \ldots, j_N) = (1, 2, \ldots, N)$ so that $|A| = \prod_{i=1}^{N} a_{ii}$ in this case. Let us call this result *Lemma 7*.

A *lower triangular matrix* $A \equiv [a_{ij}]$ is a square matrix that has zero elements above the main diagonal; i.e., $a_{ij} = 0$ for all i and j such that $1 \le i < j \le N$.

Problem 6: Show that if A is lower triangular, then $|A| \equiv \prod_{i=1}^{N} a_{ii}$. *Hint:* Use Lemmas (2) and (7).

Lemma (7) shows that it is very easy to calculate the determinant of an upper triangular matrix. Lemmas (3), (4) and (2) tell us that if we add a multiple of one row of a square matrix to another row, then the determinant of the matrix remains unchanged. This suggests an effective strategy for calculating the determinant of a square matrix A: add multiples of higher rows of A to lower rows of A in order to reduce or transform A into an upper triangular matrix $U \equiv [u_{ij}]$. Then $|A| = \prod_{i=1}^{N} u_{ii}$; i.e., the determinant of A is equal to the product of the main diagonal elements of the upper triangular matrix U.

Algorithm: Stage 1: $A \equiv [a_{ij}]$ is N by N.

Case (i): $a_{11} \neq 0$.

Add $-(a_{21}/a_{11})\, A_{1\bullet}$ to row 2 of A;
Add $-(a_{31}/a_{11})\, A_{1\bullet}$ to row 3 of A;
$\vdots$
Add $-(a_{N1}/a_{11})\, A_{1\bullet}$ to row N of A;

After doing the above row operations, the original matrix A will be transformed into a matrix that has the following form:

$$(1) \qquad \begin{bmatrix} a_{11}, & a_{12}, \ldots, a_{1N} \\ 0_{N-1}, & A^{(2)} \end{bmatrix}$$

where 0_{N-1} is an N-1 dimensional column vector of zeros and $A^{(2)}$ is an N-1 by N-1 matrix.

Case (ii): $a_{11} = 0$ but $a_{i1} \neq 0$ for some $i > 1$.

In this case, a_{11} is equal to zero but there are 1 or more nonzero elements below a_{11} in the first column of A. Let a_{i1} be the first such nonzero element in the first column of A. Simply add row i of A to the first row of A. The resulting transformed A matrix is then of the form considered in case (i) above and we can apply the algorithm outlined there to reduce the transformed A into the following form:

(2) $$\begin{bmatrix} a_{i1}, a_{12}+a_{i2}, a_{13}+a_{i3}, \ldots, a_{1N}+a_{iN} \\ 0_{N-1}, \; A^{(2)} \end{bmatrix}$$

(The N-1 by N-1 matrix $A^{(2)}$ which appears in (2) is not in general the same as the $A^{(2)}$ which appeared in (1) above).

Case (iii): $a_{i1} = 0$ for i = 1, . . ., N.

In this case, the first column of A is 0_N so the matrix already has the form given by (1) and (2) above; i.e., we have

(3) $$\begin{bmatrix} 0, & a_{12}, \ldots, a_{1N} \\ 0_{N-1}, & A^{(2)} \end{bmatrix}$$

where $A^{(2)}$ is a submatrix of the original A in this case.

Stage 2: Apply the row operations outlined in Stage 1 to the N-1 by N-1 matrix $A^{(2)}$ instead of to the N by N matrix A. Reduce $A^{(2)}$ into

$$\begin{bmatrix} a_{22}^{(3)}, a_{23}^{(3)}, a_{24}^{(3)}, \ldots, a_{2N}^{(3)} \\ 0_{N-2}, \; A^{(3)} \end{bmatrix}$$

Stage 3: Apply the row operations outlined in Stage 1 to the N-2 by N-2 matrix $A^{(3)}$ to create zeros in the elements of the first column of $A^{(3)}$ below the main diagonal.

⋮

Stage N-1: At the end of this stage, we have reduced the original A matrix into an upper triangular matrix whose determinant can readily be calculated as the product of the main diagonal elements.

Problem 7: Calculate $|A|$ if A is defined as follows:

(i) $$A \equiv \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

(ii) $$A \equiv \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Problem 8: Suppose $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Is it true that $|A+B| = |A| + |B|$?