## 4. Determinants and the Inverse of a Square Matrix

In this section, we are going to use our knowledge of determinants and their properties to derive an explicit formula for the inverse of a square matrix A provided that $|A| \neq 0$. Before we do this , we need one additional property of determinants which is a consequence of our Gaussian algorithm for computing the value of a determinant.

**Lemma 8:** Suppose the N by N matrix A has the following block upper triangular form:

$$A \equiv \begin{bmatrix} a, & b^T \\ 0_{N-1}, & C \end{bmatrix}$$

where a is a scalar, b is an N-1 dimensional column vector and C is an N-1 by N-1 matrix. Then the determinant of A is equal to a times the determinant of C; i.e.,

(4) $\quad |A| = |a| \; |C| = a \; |C|.$

*Proof:* The Guassian algorithm explained in the previous section can be used to calculate the determinant of A. For Stage 1 of the algorithm, we do not have to do anything: A is already in the form that is required at the end of Stage 1. Thus stages 2 to N-1 of the algorithm reduce the matrix C into an upper triangular matrix U say. Thus we have

(5) $$|C| = |U| = \begin{vmatrix} u_{11}, & u_{12}, & \ldots, & u_{1,N-1} \\ 0 & u_{22}, & \ldots, & u_{2,N-1} \\ \vdots & & & \\ 0, & 0, & \ldots, & u_{N-1,N-1} \end{vmatrix} = u_{11}u_{22} \ldots u_{N-1,N-1}.$$

At the same time, we have the following formula for $|A|$:

(6) $$|A| = \begin{vmatrix} a, & b_1, & b_2, & \ldots, & b_{N-1} \\ 0, & u_{11}, & u_{12}, & \ldots, & u_{1,N-1} \\ \vdots & & & & \\ 0, & 0, & 0, & \ldots, & u_{N-1,N-1} \end{vmatrix} = a\,u_{11} \ldots u_{N-1,N-1} = a\,|C|$$

where the last equality in (6) follows from (5).

Q.E.D.

**Corollary:** Suppose the N by N matrix A has the following block, lower triangular form:

(7) $$A \equiv \begin{bmatrix} a, & 0_{N-1}^T \\ b, & C \end{bmatrix}$$ Then

(8) $|A| = |a| \; |C| = a|C|$.

*Proof:* Let A be defined by (7). Then $A^T$ has the form that is required to apply Lemma (8). Thus we have, using Lemma (2),

$$|A| = |A^T| = a|C^T| \qquad \text{using Lemma (8)}$$
$$= a|C| \qquad \text{using Lemma (2)}$$

which is the desired result.

Q.E.D.

The next Lemma requires two definitions. Let A be a square N by N matrix.

**Definition:** A(i,j) denotes the ijth *minor* of the matrix A and it is the determinant of the N-1 by N-1 submatrix of A which has deleted row i and column j.

**Definition:** $A_{ij} \equiv (-1)^{i+j} A(i, j)$ denotes the ijth *cofactor* of the matrix A; it is equal to the ijth minor of A; it is equal to the ijth minor of A times minus one raised to the power i+j.

*Examples:* $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$A_{11} = (-1)^{1+1} a_{22} = a_{22}$
$A_{22} = (-1)^{2+2} a_{11} = a_{11}$
$A_{12} = (-1)^{1+2} a_{21} = -a_{21}$
$A_{21} = (-1)^{1+2} a_{12} = -a_{12}$;

$$A \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}; \; A_{12} = (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}; \; A_{13} = (-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}; \text{ etc.}$$

**Lemma (9):** *Expansion by Cofactors along the First Row:*

(9) $|A| = a_{11} A_{11} + a_{12} A_{12} + \ldots + a_{1N} A_{1N}$

*Proof*: Let $A = [a_{ij}]$ be an N by N matrix and let the N dimensional unit vectors be denoted by the columns

$$e_1 \equiv \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, e_2 \equiv \begin{bmatrix} 0 \\ 1 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_N \equiv \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}$$

It can be seen that the first row of the matrix A can be written as $\sum_{j=1}^{N} a_{1j}\, e_j^T \equiv A_{1\bullet}$. Thus we have

$$| A | = \begin{vmatrix} A_{1\bullet} \\ A_{2\bullet} \\ \vdots \\ A_{N\bullet} \end{vmatrix} = \begin{vmatrix} \sum_{j=1}^{N} a_{ij} e_j^T \\ A_{2\bullet} \\ \vdots \\ A_{N\bullet} \end{vmatrix}$$

(10) $$= \sum_{j=1}^{N} \begin{vmatrix} a_{ij} e_j^T \\ A_{2\bullet} \\ \vdots \\ A_{N\bullet} \end{vmatrix}$$ making repeated use of Lemma (4).

The first of the N determinants on the right hand side of (10) can be written as

$$\begin{vmatrix} a_{11} & 0, & \ldots, & 0 \\ \tilde{A}_{\bullet 1} & \tilde{A}_{\bullet 2}, & \ldots, & \tilde{A}_{\bullet N} \end{vmatrix} = a_{11} \left| \tilde{A}_{\bullet 2}, \ldots, \tilde{A}_{\bullet N} \right|$$

using the Corollary to Lemma (8)

(11) $$= a_{11}\, A(1, 1)$$
$$= a_{11}\, A_{11}$$

where $\tilde{A}_{\bullet j}$ is the jth column of A after dropping the first row of A, for j = 1, 2, . . ., N.

The second of the N determinants on the right hand side of (10) can be written as

$$\begin{vmatrix} 0 & a_{12} & 0 & \ldots & 0 \\ \tilde{A}_{\bullet 1} & \tilde{A}_{\bullet 2} & \tilde{A}_{\bullet 3} & \ldots & \tilde{A}_{\bullet N} \end{vmatrix} = - \begin{vmatrix} a_{12} & 0 & 0 & \ldots & 0 \\ \tilde{A}_{\bullet 2} & \tilde{A}_{\bullet 1} & \tilde{A}_{\bullet 3} & \ldots & \tilde{A}_{\bullet N} \end{vmatrix}$$

where we have interchanged the first two columns of the first matrix in the second matrix and hence by Lemma (5) we must multiply by -1 to preserve the equality

$= (-1)\, a_{12} \left| \tilde{A}_{\bullet 1}, \tilde{A}_{\bullet 3}, \ldots, \tilde{A}_{\bullet N} \right|$ by the Corollary to Lemma (8)
$= (-1)\, a_{12}\, A(1,2)$ by the definition of the minor A(1,2)
$= (-1)^{2+1}\, a_{12}\, A(1,2)$ multiplying by $(-1)^2$
$= a_{12}\, A_{12}$ by the definition of the cofactor $A_{12}$.

The third of the N determinants on the right hand side of (10) can be written as:

$$\begin{vmatrix} 0 & 0 & a_{13} & 0 & \dots & 0 \\ \tilde{A}_{\bullet 1} & \tilde{A}_{\bullet 2} & \tilde{A}_{\bullet 3} & \tilde{A}_{\bullet 4} & \dots & \tilde{A}_{\bullet N} \end{vmatrix} = (-1)^2 \begin{vmatrix} a_{13} & 0 & 0 & 0 & \dots & 0 \\ \tilde{A}_{\bullet 3} & \tilde{A}_{\bullet 1} & \tilde{A}_{\bullet 2} & \tilde{A}_{\bullet 4} & \dots & \tilde{A}_{\bullet N} \end{vmatrix}$$

where we have made 2 column interchanges to move the original third column first to column 2 and then to column 1

$= (-1)^2\, a_{13} \left| \tilde{A}_{\bullet 1}, \tilde{A}_{\bullet 2}, \tilde{A}_{\bullet 3}, \tilde{A}_{\bullet 4}, \tilde{A}_{\bullet 5}, \dots, \tilde{A}_{\bullet N} \right|$ by the Corollary to Lemma (8)
$= a_{13}\, (-1)^2\, A(1,3)$ by the definition of the minor A(1,3)
$= a_{13}\, (-1)^{3+1}\, A(1,3)$ multiplying by $(-1)^2$
$= a_{13}\, A_{13}$ by the definition of the cofactor $A_{13}$.

Continuing on in the same way for the remaining N-3 terms on the right hand side of (10), we see that formula (9) results.

Q.E.D.

**Lemma(10):** *Expansion by Cofactors along the ith Row:*

(12) $|A| = a_{i1} A_{i1} + a_{i2} A_{i2} + \dots + a_{iN} A_{iN}$ for $i = 1, 2, \dots, N$.

*Proof:* If i = 1, Lemma (10) reduces to Lemma (9). For i > 1, we make i - 1 row interchanges to move the ith row up to the first row and then we apply Lemma (9). Thus we have

$$|A| = \begin{vmatrix} A_{1\bullet} \\ \vdots \\ A_{i\bullet} \\ \vdots \\ A_{N\bullet} \end{vmatrix} = (-1)^{i-1} \begin{vmatrix} A_{i\bullet} \\ A_{1\bullet} \\ A_{2\bullet} \\ \vdots \\ A_{i-1\bullet} \\ A_{i+1\bullet} \\ \vdots \\ A_{N\bullet} \end{vmatrix}$$ because there are i - 1 row interchanges

$$= (-1)^{i-1} \left( \sum_{j=1}^{N} a_{ij} (-1)^{j+1} A(i,j) \right)$$ applying Lemma (9)

$$= \sum_{j=1}^{N} a_{ij} A_{ij}$$ using the definition of the cofactor $A_{ij}$.

Q.E.D.

**Corollary to Lemma (10):** $a_{j1} A_{i1} + a_{j2} A_{i2} + \dots + a_{jN} A_{iN} = 0$ if $i \neq j$.

*Proof:* Let $A = \begin{bmatrix} A_{1\bullet} \\ \vdots \\ A_{N\bullet} \end{bmatrix}$

Suppose we replace the ith row of the matrix A with the jth row of A where $i \neq j$. Then by lemma (1), we have

$$0 = \begin{vmatrix} A_{1\bullet} \\ \vdots \\ A_{j\bullet} \\ \vdots \\ A_{j\bullet} \\ \vdots \\ A_{N\bullet} \end{vmatrix} \leftarrow \text{ ith row has been replaced by jth row.}$$

$= a_{j1} A_{i1} + a_{j2} A_{i2} + \ldots + a_{jN} A_{iN}$ using lemma (10) except that the numbers $a_{i1}, a_{i2}, \ldots, a_{iN}$ have been replaced by $a_{j1}, a_{j2}, \ldots, a_{jN}$.

Q.E.D.

Lemma (10) and its corollary enable us to calculate a *right inverse* of an N by N matrix A provided that $|A| \neq 0$.

**Lemma (11):** If A is an N by N matrix and $|A| \neq 0$, then a right inverse for A, say $A_R^{-1}$, given by

$$A_R^{-1} = \begin{bmatrix} \frac{A_{11}}{|A|} & \ldots & \frac{A_{N1}}{|A|} \\ \vdots & & \\ \frac{A_{1N}}{|A|} & \ldots & \frac{A_{NN}}{|A|} \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1N} \\ A_{21} & A_{22} & \ldots & A_{2N} \\ \vdots & & & \\ A_{N1} & A_{N2} & \ldots & A_{NN} \end{bmatrix}^T$$

where $A_{ij}$ is the ijth cofactor of the matrix A.

*Proof:* Take the ith row of A, $A_{i\bullet}$, time the jth column of $A_R^{-1}$. We get

$$a_{i1}\frac{A_{j1}}{|A|} + a_{i2}\frac{A_{j2}}{|A|} + \ldots + a_{iN}\frac{A_{jN}}{|A|} = \begin{cases} \frac{|A|}{|A|} & \text{if } i = j \text{ by lemma (10)} \\ \frac{0}{|A|} & \text{if } i \neq j \text{ by corollary to (10)} \end{cases}$$

Thus we have $A\,A_R^{-1} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & 1 \end{bmatrix} = I_N$ and thus $A_R^{-1}$ is a right inverse for A by definition.

Q.E.D.

**Problem 9:** Calculate a right inverse for the following matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{assuming } ad - bc \neq 0 \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 3 & 4 \end{bmatrix}.$$

**Problem 10:** Suppose that the N by N matrix A has a right inverse B and a left inverse C. Show that B = C. *Hint*: You will need the results of problem 2.

**Problem 11:** If A is an N by N matrix and $|A| \neq 0$, show that a *left* inverse *exists*. (Note: Problem *assumed* the existence of a left inverse; here we have to show that one exists. *Hint*: If $|A| \neq 0$, then $|A^T| \neq 0$. Thus $A^T$ will have a right inverse by lemma (11)).

**Problem 12:** If A is an N by N matrix and $|A| = 0$, show that there does not exist an N by N matrix B such that $AB = I_N$. *Hint:* You may find lemma (6) useful.

The above results give us an easily? checked condition for the existence of an inverse matrix for an N by N matrix A: namely if $|A| \neq 0$, then a common right and left inverse matrix exists which we will denote by $A^{-1}$; If $|A| = 0$, then $A^{-1}$ does not exist. There is another convenient condition on an N by N matrix A which will ensure that $A^{-1}$ exists and we will develop it below after we discuss Cramer's Rule.

## 5. Expansion by Cofactors along a column and Cramer's Rule.

Applying lemma (10) to $A^T$ yields the following equation:

(13) $|A^T| = a_{1i} A_{1i} + a_{2i} A_{2i} + \dots + a_{Ni} A_{Ni}$ for any *column* index $i = 1, \dots, N$

(i.e., we have simply interchanged row and column indices)

$= |A|$ since $|A| = |A^T|$.

Now let us consider the following system of N simultaneous equations in N unknowns $x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$:

(14) $Ax = b$ where

A = N by matrix of coefficients

$b = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix}$ vector of constants.

If $|A| \neq 0$, the solution to (14) is given by

(15) $A^{-1} Ax = A^{-1} b$

or $Ix = A^{-1} b$
or $x^* = A^{-1} b$

$$= \frac{\begin{bmatrix} A_{11} & \ldots & A_{1N} \\ A_{21} & \ldots & A_{2N} \\ \vdots & & \\ A_{N1} & \ldots & A_{NN} \end{bmatrix}^T b}{|A|}$$

using lemma (11) (and problems 10 and 11; i.e. that left and right inverses exist and coincide.)

(16) Therefore, $x_i^*$ = ith component of x

$$= \frac{b_1 A_{1i} + b_2 A_{2i} + \ldots + b_N A_{Ni}}{|A|}$$

(perform the relevant matrix multiplication)

$$= \frac{|A_{\bullet 1}, A_{\bullet 2}, \ldots, A_{\bullet i-1}, b, A_{\bullet i+1}, \ldots, A_{\bullet N}|}{|A|}$$

Using (13) with the vector b replacing the ith column of A for i = 1, 2, . . ., N.

That is $x_i$ may be found by replacing the ith column of the matrix A by the column vector b, take the determinant of the resulting matrix and divide by the determinant of the original matrix A. Result (16) is known as *Cramer's Rule*.

**Problem 13:** Given Ax = b where $A \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 3 & 4 \end{bmatrix}$ and $b = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$, calculate the solution $x^*$.

**6. The Gauss Elimination Method for Constructing $A^{-1}$**

Define the following two *elementary row operations*:

(i) add a scalar times a row of a matrix to another row of a matrix and

(ii) multiply a row of a matrix by a *nonzero* scalar.

The above two types of elementary row operations can be applied to a square matrix A in order to determine whether $A^{-1}$ exists and if $A^{-1}$ exists, these operations can be used to construct an effective method for actually constructing $A^{-1}$.

Recall the Gaussian algorithm in section 3 above which used elementary row operations of type (i) above to reduce A down to an upper triangular matrix U. If any of the diagonal elements of U are equal to zero; i.e., we have $u_{ii} = 0$ for some i, then

$$|A| = \prod_{i=1}^{N} u_{ii} = 0$$

and the results in the previous section tell us that $A^{-1}$ cannot exist.

However, if all of the $u_{ii} \neq 0$, then we can continue to use elementary row operations of the first type to further reduce U into a diagonal matrix, say

$$(17) \quad D = \begin{bmatrix} u_{11} & 0 & \dots & 0 \\ 0 & u_{22} & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & u_{NN} \end{bmatrix}$$

Finally, once A has been reduced to the form (17), we can apply type (ii) elementary row operations and reduce the diagonal matrix to the N by N identity matrix $I_N$.

Each of the two types of elementary row operations can be represented by premultiplying A by an N by N matrix. The operation of adding k times row i of A to row j can be accomplished by premultiplying A by the following matrix:

(18) $E = I_N + k\, e_j\, e_i^T$; $i \neq j$.

Note that E is lower triangular if i < j and is upper triangular if i > j. In either case, $|E| = 1$.

The operation of multiplying the ith row of A by $k \neq 0$ can be accomplished by premultiplying A by the diagonal matrix

(19) $$D \equiv \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & & \\ 0 & 0, k & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & 1 \end{bmatrix}$$ ith row; i.e., $d_{jj} = 1$ if $i \neq j$ and $d_{ii} = k$.

Note that the determinant of D equals $k \neq 0$.

In the case where $|A| \neq 0$, we can premultiply A by a series of elementary row matrices of the form (18) and (19), say $E_n, E_{n-1}, \ldots, E_1$ such that the transformed A is reduced to the identity matrix; i.e., we have

(20) $E_n, E_{n-1}, \ldots, E_1\, A = I_N$

Thus $B \equiv (E_n, E_{n-1}, \ldots, E_1)$ is a left inverse for A by the definition of a left inverse.

To construct B, we need only apply the elementary row operation matrices $E_n$, $E_{n-1}, \ldots, E_1$ to $I_N$: i.e.,

(21) $B = A^{-1} \equiv E_n, E_{n-1}, \ldots, E_1\ I_N$.

Thus as we reduce A to $I_N$ by means of elementary row operations, apply the same elementary row operations to $I_N$ and in the end, $I_N$ will be transformed into $A^{-1}$.

*Example:*

$$A \equiv \begin{bmatrix} 1, & 2 \\ 3, & 4 \end{bmatrix} \quad I_2 \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Take -3 times row 1 and add to row 2; get:

$$\begin{bmatrix} 1, & 2 \\ 0, & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

Multiply the second row by -1/2; get:

$$\begin{bmatrix} 1, & 2 \\ 0, & 1 \end{bmatrix} \qquad \begin{bmatrix} 1, & 0 \\ \frac{3}{2}, & -\frac{1}{2} \end{bmatrix}.$$

Now add -2 times row 2 to row 1; get:

$$\begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix} \qquad \begin{bmatrix} -2, & 1 \\ \frac{3}{2}, & -\frac{1}{2} \end{bmatrix} = A^{-1}.$$

*Check:* $A \bullet A^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -2+3 & 1-1 \\ -6+6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$

Now each elementary operation can be represented by means of a matrix; i.e., the first elementary row operation can be represented by the matrix $E_1$:

$$E_1 \equiv \begin{bmatrix} 1, & 0 \\ -3, & 1 \end{bmatrix} \text{ and } E_1\ A = \begin{bmatrix} 1, & 0 \\ -3, & 1 \end{bmatrix} \begin{bmatrix} 1, & 2 \\ 3, & 4 \end{bmatrix} = \begin{bmatrix} 1, & 2 \\ 0, & -2 \end{bmatrix}$$

The second elementary row operation matrix is $E_2$:

$$E_2 \equiv \begin{bmatrix} 1, & 0 \\ 0, & -\frac{1}{2} \end{bmatrix} \text{ and } E_2\ (E_1A) = \begin{bmatrix} 1, & 0 \\ 0, & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1, & 2 \\ 0, & -2 \end{bmatrix} = \begin{bmatrix} 1, & 2 \\ 0, & 1 \end{bmatrix}$$

The final elementary row operation matrix is $E_3$:

$$E_3 \equiv \begin{bmatrix} 1, & -2 \\ 0, & 1 \end{bmatrix} \text{ and } E_3\ (E_2E_1A) = \begin{bmatrix} 1, & -2 \\ 0, & 1 \end{bmatrix} \begin{bmatrix} 1, & 2 \\ 0, & 1 \end{bmatrix} = \begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix}$$

Thus we have $E_3(E_2(E_1\ A))) = (E_3\ E_2\ E_1)A = I_2$. Thus $E_3\ E_2\ E_1$ is a left inverse for A and by the results in the previous section is also the unique right inverse.

**Problem 13:** Let A be a 2 by N matrix. Find a sequence of elementary row operations of the form defined by (i) and (ii) above that will interchange the rows of A; i.e., transform $A \equiv \begin{bmatrix} A_{1\bullet} \\ A_{2\bullet} \end{bmatrix}$ into $\begin{bmatrix} A_{2\bullet} \\ A_{1\bullet} \end{bmatrix}$ using the two elementary row operations that we have defined. *Hint:* four elementary row operations will be required.