7. Linear Independence and the Rank of a Matrix

The results in this section provide another condition which is necessary and sufficient for the existence of the inverse for a square matrix.

Let $A \equiv [A_{\bullet 1}, A_{\bullet 2}, \ldots, A_{\bullet N}]$ be an M by N matrix. We say that the N column vectors of A are *linearly independent* if the only solution x to

(22) $Ax = 0_M$

is $x = 0_N$. If a solution vector $x \neq 0_N$ to (22) exists, then we say that the columns of A are *linearly dependent*.

How can we determine whether the columns of A are linearly dependent or independent? The Gaussian triangularization algorithm developed in section 3 above can be used to answer this question.

Consider *Stage 1* of the Guassian algorithm. If we end up in case (iii) (so that $A_{1\bullet} = 0_M$), then we can satisfy (22) by choosing $x = e_1$ (where $e_1 \equiv (1, 0_{M-1}^T)^T$ is the first unit vector of dimension M). In this case where $A_{1\bullet} = 0_M$, we can immediately deduce that the columns of A are linearly dependent.

Now assume that cases (i) or (ii) occurred in Stage 1 of the algorithm and we move on to *Stage 2* (assuming N and M are greater than one) of the algorithm. If case (iii) occurs in Stage 2, then at the end of Stage 2, the first two columns of the transformed A matrix have the following form:

(23) $$\begin{bmatrix} u_{11}, & u_{12} \\ 0_{M-1}, & 0_{M-1} \end{bmatrix}$$

where $u_{11} \neq 0$. Consider solving the following equation:

(24) $u_{11} x_1 + u_{12} x_2 = 0$.

If we set $x_2 = 1$, then since $u_{11} \neq 0$, we can solve (24) for x_1 as follows:

(25) $x_1 = -u_{12}/u_{11}$.

Let the M by M matrix E denote the product of the elementary row operation matrices that transform the first two columns of A into the case (iii) upper triangular matrix defined by (23). Now premultiply both sides of (22) by E to obtain:

(26) $EAx = E0_M = 0_M$.

It can be seen, using (23) - (25), that if we choose x to be the following vector:

(27) $x^* \equiv -(u_{12}/u_{11})e_1 + e_2 \neq 0_N$,

then x^* satisfies (26). Recall from the previous section that each elementary row matrix that adds a multiple of one row to another row has a determinant equal to one. Since E is a product of these matrices, its determinant will also equal one. Hence E^{-1} exists and we can premultiply both sides of $EAx^* = 0_M$ by E^{-1} and conclude that $Ax^* = 0_M$ with $x^* \neq 0_N$. Thus if case (iii) occurs at the end of Stage 2 of the Gaussian triangularization algorithm, we can conclude that the columns of A are linearly dependent.

We now need to consider two cases dependent on whether the number of rows of A (M) is greater or less than the number of columns of A(N).

Case (1): $M \geq N$.

In this case, we follow the Gaussian algorithm through all N stages. If at the end of any stage (say stage i) of the algorithm, we find that $u_{ii} = 0$, we can adapt the above stage 2 argument to show that there is a nonzero x^* vector (which has $x_i^* = 1$ and $x_j^* = 0$ for $j > i$) such that $Ax^* = 0_M$ and hence the columns of A are linearly dependent.

On the other hand, if *all* of the diagonal elements of the final upper triangular matrix are nonzero, then we can show that the columns of A are linearly independent. In this case, the final U matrix has the following form:

(28) $$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1N} \\ 0 & u_{22} & \cdots & u_{2N} \\ \vdots & & & \\ \vdots & & & \\ 0 & 0 & \cdots & u_{NN} \\ 0_{M-N} & 0_{M-N} & \cdots & 0_{M-N} \end{bmatrix}, \qquad \prod_{i=1}^{N} u_{ii} \neq 0.$$

Let E represent the product of the elementary row matrices that transform A into the U defined by (28); i.e., we have

(29) $EA = U \quad ; \; |E| = 1.$

Premultiply both sides of (22), $Ax = 0_M$, by E to obtain:

(30) $EAx = Ux = E0_M = 0_M$.

Using (28), we see that the Nth equation in (31) is:

(31) $u_{NN}\, x_N = 0$

and since $u_{NN} \neq 0$ by hypothesis, we must have $x_N = 0$. Now look at the N-1 st equation in (30):

(32) $\quad u_{N-1,N-1}\, x_{N-1} + u_{N-1,N}\, x = 0.$

Substituting $x_N = 0$ into (32) yields

(33) $\quad u_{N-1,N-1}\, x_{N-1} = 0$

and since $u_{N-1,N-1} \neq 0$ by hypothesis, we must have $x_{N-1} = 0$. Continuing on in the same way, we deduce that the only x solution to (30) is $x^* = 0_N$.

It is obvious that $x^* = 0_N$ satisfies $Ax = 0_M$. Could there be any other solution to $Ax = 0_M$? Let x^{**} be such that

(34) $\quad Ax^{**} = 0_M.$

Premultiplying both sides of (34) by E leads to:

(35) $\quad EAx^{**} = Ux^{**} = 0_M.$

But the only solution to (35) is $x^{**} = 0_N$. Hence under our Case (1) hypothesis where $M \geq N$ and all $u_{ii} \neq 0$, $i = 1, 2, \ldots, N$, we deduce that the columns of A are linearly independent. If any of the $u_{ii} = 0$, then the columns of A are linearly dependent.

Case 2: $M < N$.

In this case, carry out the Gaussian triangularization procedure until we run out of rows. The final U matrix will have the following form:

(36) $$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1M} & u_{1M+1} & \cdots & u_{1N} \\ 0 & u_{22} & \cdots & & u_{2M+1} & \cdots & u_{2N} \\ \vdots & & & u_{2M} & & & \\ 0 & 0 & \cdots & u_{MM} & u_{MM+1} & \cdots & u_{MN} \end{bmatrix}.$$

If any of the $u_{ii} = 0$, then we can adapt our previous arguments to show that the columns of A are linearly dependent. For example, suppose u_{22} is the first zero u_{ii}. Then the $x^* \neq 0_N$ defined by (27) will satisfy $Ax^* = 0_M$.

If $u_{ii} \neq 0$ for $i = 1, 2, \ldots, M$, then consider the equations $Ux = 0_M$. If we set $x^*_{M+1} = -1$ and $x^*_{M+2} = x^*_{M+3} = \ldots = x^*_N = 0$, then the equations $Ux = 0_N$ reduce to

$$(37)\quad \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1M} \\ 0 & u_{22} & \cdots & u_{2M} \\ \vdots & & & \\ 0 & 0 & & u_{MM} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{bmatrix} = \begin{bmatrix} u_{1M+1} \\ u_{2M+1} \\ \vdots \\ u_{MM+1} \end{bmatrix}$$

which can readily be solved for $x_1^*, \ldots, x_M^*$; i.e.,

$$(38)\quad x_M^* = u_{MM+1} / u_{MM};$$

$$x_{M-1}^* = [u_{M-1,M+1} - u_{M-1,M} x_M^*] / u_{M-1,M-1}; \text{ etc.}$$

The resulting $x^* \neq 0_N$ and hence we deduce that the columns of A are linearly dependent.

Thus if we are in Case (2), we inevitably deduce that the columns of A are linearly dependent.

Putting all of the above material together, we find that the columns of A are linearly dependent *unless* $M \geq N$ and the N u_{ii} elements in (28) are all nonzero. Only in this last case, are the columns of A linearly independent.

Definition: The rank of an N by M matrix is the maximal number of linearly independent columns which it contains.

Example the rank of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is 3, the rank of

$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ is also 3.

Lemma (11): If the rank of the N by N matrix A is N, then A^{-1} exists.

Proof: If the rank of the N by N matrix is N, then all of the columns of A are linearly independent. Hence, when implementing the Gaussian triangularization of A, all of the diagonal elements u_{ii} of the upper triangular matrix U must be nonzero. Hence the determinant of $U (= \prod_{i=1}^{N} u_{ii})$ is also nonzero. Recall that

$$(39)\quad EA = U \qquad \text{where} \qquad |E| = 1.$$

Hence, taking determinants on both sides of (39):

$$(40)\quad |EA| = |E|\ |A| = |A| = |U| = \prod_{i=1}^{N} u_{ii} \neq 0,$$

and we conclude that $|A| \neq 0$ so A^{-1} exists.

Q.E.D.

Problem 14: Let A by M by N where M > N and consider the system of equations

(i) $Ax = b$

where x is an N dimensional solution vector and b is an M dimensional vector of parameters. Suppose the N columns of $A \equiv [A_{\bullet 1}, A_{\bullet 2}, \ldots, A_{\bullet N}]$ are linearly independent. Under what conditions on b will a solution x to (i) exist and how could you compute it if it did exist? *Hint:* Make use of the M by M elementary row matrix E which reduces A to upper triangular form U; i.e., E and U satisfy (28) and (29) in the text above.

8. Comparative Statics Analysis of a System of Linear Equations

Let A be an N by N matrix and b an N dimensional vector. If $|A| \neq 0$, then the solution x to $Ax = b$ can be written as:

(41) $x = A^{-1} b.$

Obviously, the components of the solution vector x depend on the components a_{ij} of A and b_i of $b \equiv [b_1, b_2, \ldots, b_N]^T$. How does x change as the a_{ij} and b_i change?

Using (41), the N by N matrix of the derivatives of x_i with respect to b_j, $\partial x_i/\partial b_j$, can be written as

(42) $\nabla_b x = [\partial x_i/\partial b_j] = A^{-1}.$

Recalling the determinantal formula for A^{-1} given in Lemma (11), we see that

(43) $\partial x_i/\partial b_j = A_{ji}/|A|$; $1 \leq i, j \leq N$

where A_{ji} is the jith cofactor of A.

In order to determine how the components of x change as the components of A change, it is convenient to study a somewhat more general problem: we let *each* component of the A matrix be a function of the scalar variable t (i.e., $a_{ij} = a_{ij}(t)$ for $1 \leq i, j \leq N$) and then x defined by (41) will also be a function of t, x(t). We then compute the vector of derivatives, $x'(t) \equiv [x_1'(t), \ldots, x_N'(t)]^T$. Before we do this, we establish a preliminary result.

Lemma (12): Let $A(t) \equiv [a_{ij}(t)]$ have N columns and $B(t) \equiv [b_{ij}(t)]$ have N rows so that $C(t) \equiv A(t)B(t)$ is well defined. Note that each element of A(t) and each element of B(t) is a function of the scalar variable t. Then the matrix of derivatives with respect to t of the product matrix is

(44) $C'(t) = A(t)B'(t) + A'(t)B(t)$

where $A'(t) \equiv [a'_{ij}(t)]$ and $B'(t) \equiv [b'_{ij}(t)]$ are the matrices of derivatives of A(t) and B(t).

Proof: The ijth element of C is

(45) $c_{ij}(t) = A_{i\bullet}(t)B_{\bullet j}(t) = \sum_{n=1}^{N} a_{in}(t)b_{nj}(t).$

Differentiating (45) with respect to t yields for all i and j:

(46) $c'_{ij}(t) = \sum_{n=1}^{N} a_{in}(t)b'_{nj}(t) + \sum_{n=1}^{N} a'_{in}(t)b_{nj}(t).$

It can be seen that equations (46) are equivalent to equations (44).

Q.E.D.

Now let the B(t) matrix which appears in Lemma (12) be $A^{-1}(t)$ and differentiate both sides of the following identity with respect to t:

(47) $A(t)A^{-1}(t) = I_N.$

Using Lemma (12), we obtain:

(48) $A'(t)A^{-1}(t) + A(t)[dA^{-1}(t)/dt] = 0_{N \times N}$

where $dA^{-1}(t)/dt \equiv [da_{ij}^{-1}(t)/dt]$ is the N by N matrix of derivatives of the components of A^{-1} with respect to t. Premultiply both sides of (48) by $A^{-1}(t)$ and after rearranging terms, we obtain the following formula:

(49) $dA^{-1}(t)/dt = -A^{-1}(t)A'(t)A^{-1}(t).$

Now return to (41) which we rewrite as:

(50) $x(t) = A^{-1}(t)b.$

Differentiating (50) with respect to t and using (49) yields:

(51) $x'(t) \equiv [x'_1(t), \ldots, x'_N(t)]^T = -A^{-1}(t)A'(t)A^{-1}(t)b.$

If only a_{ij} depends on t, then

(52) $A'(t) = e_i e_j^T a'_{ij}(t)$

where e_i and e_j are the i and jth unit vectors. Substituting (52) into (51) yields in this special case:

(53) $x'(t) = -A^{-1}(t) e_i a'_{ij}(t) e_j^T A^{-1}(t) b.$

Problem 15: Suppose the N components of the b vector are all functions of the scalar variable t; i.e., we have $b(t) \equiv [b_1(t), \ldots, b_N(t)]^T$. Define

(i) $x(t) = A^{-1} b(t)$

where the N by N matrix A does not depend on t and $|A| \neq 0$. Exhibit a formula for the vector of derivatives $x'(t)$. *Hint:* This problem is easy!

Problem 16: Let A be an N by N matrix. Regard $|A|$ as a function of the ijth element of A, a_{ij}; i.e., define the function $f(a_{ij}) = |A|$. Find a formula for the derivatives of the determinant of A with respect to a_{ij}; i.e., calculate $f'(a_{ij})$. *Hint:* use Lemma (10).