Problem:

4. Define $E = E_3E_2E_1$ where E_3 is defined by (62) and E_1 and E_2 are defined in (61). Show that $EAE^T = D$ where D is defined by (60).

The matrix E and the diagonal matrix D which occurs in the Lagrange-Gauss diagonalization procedure (see (59) above) can be used to determine whether the symmetric A satisfies any of the definiteness properties (48) - (52).

Consider the E matrix which occurs in (59). Since E is a product of elementary row matrices, each of which has determinant equal to 1, it can be seen that

(63) $|E| = |E^T| = 1.$

Since $|E^T| = 1$, $(E^T)^{-1}$ exists. Now for each $x \neq 0_N$, consider the y defined by

(64) $y = (E^T)^{-1} x.$

Suppose $y = 0_N$. Then premultiplying both sides of (64) by E^T leads to $x = 0_N$ which contradicts $x \neq 0_N$. Hence if $x \neq 0_N$, then the y defined by (64) also satisfies $y \neq 0_N$.

Let $x \neq 0_N$ and define y by (64). Premultiplying both sides of (64) by E^T leads to

(65) $x = E^Ty$ where $y \neq 0_N$.

Hence for $x \neq 0_N$, we have

$$
\begin{aligned}
x^TAx &= (E^Ty)^TA(E^Ty) && \text{using (65)} \\
&= y^TEAE^Ty \\
&= y^TDy && \text{using (59)} \\
&= \textstyle\sum_{i=1}^{N} d_{ii}\, y_i^2. && (66)
\end{aligned}
$$

Thus necessary and sufficient conditions for A to be *positive definite* are:

(67) $d_{ii} > 0$ for $i = 1, 2, \ldots, N$.

Using (66) and (49), it can be seen that necessary and sufficient conditions for A to be *negative definite* are:

(68) $d_{ii} < 0$; $i = 1, \ldots, N$.

Similarly, necessary and sufficient conditions for A to be *positive semidefinite* are:

(69) $d_{ii} \geq 0$; $i = 1, \ldots, N$.

Finally, necessary and sufficient conditions for A to be *negative semidefinite* are:

(70) $d_{ii} \leq 0$; $i = 1, \ldots, N$.

Problem:

5. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$. Which of the definiteness properties (48) - (52) does A satisfy?

Historical Note: The above reduction of a quadratic form x^TAx to a sum of squares y^TDy was accomplished by J.-L. Lagrange (1759), "Researches sur la métode de moximis et miniouis", *Miscellanea Taurinensi*, 1, for the cases N = 2 and N = 3. Carl Friedrich Gauss described the general algorithm in 1810; see his *Theory of the Combination of Observations Least Subject to Errors*, G.W. Stewart, translator, SIAM Classics in Applied Mathematics, 1995. This publication indicates that Gauss arrived at the principle of least squares estimation in 1794 or 1795 but the French mathematician A.M. Legendre independently derived the principle (and named it) in 1805 and actually published the method before Gauss.

6. Checking Second Order Conditions Using Determinants

Let $A = [a_{ij}]$ be an N by N symmetric matrix and suppose that we want to check whether A is a positive definite matrix.

If A is positive definite, then it must be the case that $a_{11} > 0$. Why is this?

By the definition of A being positive definite, (48) above, we must have $x^TAx > 0$ for all $x \neq 0_N$. Let $x = e_1$, the first unit vector. Then if A is positive definite, we must have

(71) $e_1^T A e_1 = a_{11} > 0$.

We can rewrite (71) using determinantal notation. Since the determinant of a one by one matrix is simply equal to the single element, (71) is equivalent to:

(72) $|a_{11}| > 0$.

Now if the N by N matrix A is positive definite, it can be seen that we must have

(73) $0 < [x_1, x_2, 0_{N-2}^T]A[x_1, x_2, 0_{N-2}^T]^T = [x_1, x_2]\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

for all x_1, x_2 such that $[x_1, x_2] \neq [0, 0]$. This means that the top left corner 2 by 2 submatrix of A must also be positive definite if A is positive definite. Hence, by the previous section, there exists a 2 by 2 elementary row matrix $E^{(2)}$ which, along with $E^{(2)T}$, reduces the 2 by 2 submatrix of A into diagonal form. Using (71), it can be seen that the $E^{(2)}$ which will do the job is

(74) $$E^{(2)} \equiv \begin{bmatrix} 1, & 0 \\ -a_{12}/a_{11}, & 1 \end{bmatrix}$$

and we have

(75) $$E^{(2)} \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} E^{(2)T} = \begin{bmatrix} d_{11}, & 0 \\ 0, & d_{22} \end{bmatrix}$$

where the d_{ii} turn out to be:

(76) $d_{11} \equiv a_{11}$;

(77) $d_{22} \equiv a_{22} - a_{12}^2 / a_{11}$.

From the previous section, we know that necessary and sufficient conditions for the 2 by 2 submatrix of A to be positive definite are:

(78) $d_{11} > 0$; $d_{22} > 0$.

Since $|E^{(2)}| = 1$, taking determinants on both sides of (75) yields:

(79) $$\begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} = \begin{vmatrix} d_{11} & 0 \\ 0 & d_{22} \end{vmatrix} = d_{11}d_{22} > 0$$

where the inequality follows from (78).

Using (76) and (79), it can be seen that the determinantal conditions:

(80) $|a_{11}| > 0$;

(81) $$\begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} > 0$$

are necessary and sufficient for conditions (78) which in turn are necessary and sufficient for the positive definiteness of the top left corner 2 by 2 submatrix of A.

If the N by N matrix A is positive definite, it can be seen that we must have

(82) $$0 < [x_1, x_2, x_3, 0_{N-3}^T] A [x_1, x_2, x_3, 0_{N-3}^T]^T$$

$$= [x_1, x_2, x_3] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

for all $[x_1, x_2, x_3] \neq [0, 0, 0]$. This means that the top left corner 3 by 3 submatrix of A must also be positive definite. Hence there exists a 3 by 3 elementary row matrix $E^{(3)}$ with $|E^{(3)}| = 1$ such that

(83) $$E^{(3)} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} E^{(3)T} = \begin{bmatrix} d_{11}^{(3)}, & 0, & 0 \\ 0, & d_{22}^{(3)}, & 0 \\ 0, & 0, & d_{33}^{(3)}, \end{bmatrix}$$

where the $d_{ii}^{(3)}$ satisfy:

(84) $$d_{11}^{(3)} > 0;\ d_{22}^{(3)} > 0;\ d_{33}^{(3)} > 0.$$

Since $|E^{(3)}| = 1$, taking determinants on both sides of (83) yields

(85) $$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = \begin{vmatrix} d_{11}^{(3)}, & 0, & 0 \\ 0, & d_{22}^{(3)}, & 0 \\ 0, & 0, & d_{33}^{(3)} \end{vmatrix} = d_{11}^{(3)} d_{22}^{(3)} d_{33}^{(3)} > 0$$

where the inequality in (85) follows from (84).

When A is positive definite, we need to show that the $d_{11}^{(3)}$ and $d_{22}^{(3)}$ which occur in (83) - (85) are the same as the d_{11} and d_{22} which occurred in (76) - (79). But this is obviously true using the Gaussian diagonalization algorithm: when we diagonalize the 3 by 3 submatrix of A, we must first diagonalize the 2 by 2 submatrix of A and hence the $d_{11}^{(3)}$ and $d_{22}^{(3)}$ in (83) will equal the d_{11} and d_{22} which occurred in (75). Hence, we can rewrite (85) as follows:

(86) $$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = \begin{vmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{vmatrix} = d_{11} d_{22} d_{33} > 0;$$

i.e., we have dropped the superscripts on the d_{ii}. Now it can be seen that the determinantal inequalities (80), (81) and

$$(87) \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} > 0$$

along with the equalities in (76), (79) and (86) are necessary and sufficient for the inequalities

$$(88) \qquad d_{11} > 0, d_{22} > 0, d_{33} > 0$$

which in turn are necessary and sufficient to for the top left 3 by 3 submatrix of A to be positive definite.

Obviously, the above process can be continued until we obtain the following N determinantal conditions which are necessary and sufficient for the N by N symmetric matrix A to be *positive definite*:

$$(89) \qquad |a_{11}| > 0; \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} > 0; \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} > 0; \ldots; |A| > 0.$$

How can we adapt the above analysis to obtain conditions for A to be *negative definite*? Obviously, the Gaussian diagonalization procedure can again be used: the only difference in the analysis will be that the diagonal elements d_{ii} must all be *negative* in the case where A is negative definite. This means that the determinantal conditions in (89) that involve an *odd* number of rows and columns of A must have their signs changed, since these determinants will equal the product of an *odd* number of the d_{ii}. Hence the following N determinantal conditions are necessary and sufficient for the N by N symmetric matrix A to be *negative definite*:

$$(90) \qquad |a_{11}| < 0; \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} > 0; \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} < 0; \ldots; (-1)^N |A| > 0.$$

Turning now to determinantal conditions for positive semidefiniteness or negative seimdefiniteness, one might think that the conditions are a straightforward modification of conditions (89) and (90) respectively, where the strict inequalities (>) are replaced by weak inequalities (≥). Unfortunately, this thought is incorrect as the following example shows.

Example: $A \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$

In this case, we see that

$|a_{11}| = |0| = 0;$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 \text{ and}$$

$|A| = 0$. Hence the weak inequality form of conditions (89) and (90) are both satisfied so we might want to conclude that this A is both positive and negative semidefinite. However, this is not so: A is indefinite since $e_2^T A e_2 = a_{22} = 1 > 0$ and $e_3^T A e_3 = a_{33} = -1 < 0$.

The problem with this example is that all of the elements in the first row and column of A are zero and hence d_{11} is zero. Now look back at the inequalities (79) and (86): it can be seen that if $d_{11} = 0$, then these inequalities are no longer valid. However, if instead of always picking submatrices of A that included the first row and column of A, we picked submatrices of A that excluded the first row and column, then we would discover that the submatrix of A which consisted of rows 2 and 3 and columns 2 and 3 is indefinite; i.e., we have

(91) $\quad a_{22} = 1 > 0$ and $\begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix} = -1 < 0.$

In order to determine whether A is positive semidefinite, we replace the strict inequalities in (89) by weak inequalities but the resulting weak inequalities must be checked for all possible choices of the rows of A; i.e., necessary and sufficient conditions for A to be *positive semidefinite* are:

(92) $\quad |a_{ii}| = a_{ii} \geq 0 \quad$ for $\quad i = 1, 2, \ldots, N;$

$$\begin{vmatrix} a_{i_1 i_1} & a_{i_1 i_2} \\ a_{i_1 i_2} & a_{i_2 i_2} \end{vmatrix} \geq 0 \quad \text{for} \quad i \leq i_1 < i_2 \leq N;$$

$$\begin{vmatrix} a_{i_1 i_1} & a_{i_1 i_2} & a_{i_1 i_3} \\ a_{i_1 i_2} & a_{i_2 i_2} & a_{i_2 i_3} \\ a_{i_1 i_3} & a_{i_2 i_3} & a_{i_3 i_3} \end{vmatrix} \geq 0 \quad \text{for} \quad 1 \leq i_1 < i_2 < i_3 \leq N;$$

$$\vdots$$

$$|A| \geq 0.$$

In the 2 by 2 case, conditions (92) boil down to the following 3 conditions:

(93) $\quad a_{11} \geq 0;\ a_{22} \geq 0;\ |A| = a_{11}a_{22} - a_{12}^2 \geq 0.$

In the 3 by 3 case, conditions (92) reduce to the following 7 determinantal conditions:

(94) $a_{11} \geq 0$; $a_{22} \geq 0$; $a_{33} \geq 0$;

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} \geq 0; \begin{vmatrix} a_{11} & a_{13} \\ a_{13} & a_{33} \end{vmatrix} \geq 0; \begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix} \geq 0; \ |A| \geq 0.$$

If A is a symmetric N by N matrix, then necessary and sufficient determinantal conditions for A to be *negative semidefinite* are:

(95) $(-1)\ |a_{ii}| \geq 0;$ $\qquad i = 1, 2, \ldots, N;$

$$(-1)^2 \begin{vmatrix} a_{i_1 i_1} & a_{i_1 i_2} \\ a_{i_1 i_2} & a_{i_2 i_2} \end{vmatrix} \geq 0; \qquad 1 \leq i_1 < i_2 \leq N;$$

$$(-1)^3 \begin{vmatrix} a_{i_1 i_1} & a_{i_1 i_2} & a_{i_1 i_3} \\ a_{i_1 i_2} & a_{i_2 i_2} & a_{i_2 i_3} \\ a_{i_1 i_3} & a_{i_2 i_3} & a_{i_3 i_3} \end{vmatrix} \geq 0; \qquad 1 \leq i_1 < i_2 < i_3 \leq N;$$

$$\vdots$$

$$(-1)^N\ |A| \geq 0.$$

Problems:

6. Let $A \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Use the Gaussian diagonalization procedure to determine the definiteness properties of A.

7. Does the A defined in problem 6 above satisfy the determinantal conditions (93) for positive semidefiniteness?

8. Solve $\max_{x_1, x_2} \{f(x_1, x_2)\colon\ x_1 > 0, x_2 > 0\}$ (if possible) where f is defined as follows:

(a) $f(x_1, x_2) \equiv -x_1^2 + x_1 x_2 - x_2^2 + x_1 + x_2$

(b) $f(x_1, x_2) \equiv \ln x_1 + \ln x_2 + x_1 x_2 - 2x_1 - 2x_2$.

Check second order conditions when appropriate.

9. Consider the following 2 input, 1 output profit maximization problem:

(i) $\max_{y, x_1, x_2} \{py - w_1 x_1 - w_2 x_2 :\ y = f(x_1, x_2)\}$

where f is the producer's production function, $w_i > 0$ is the price of input i and $p > 0$ is the price of output. The unconstrained maximization problem that is equivalent to (i) is:

(ii) $\max_{x_1, x_2} \{ pf(x_1, x_2) - w_1x_1 - w_2x_2 \}$.

Assume that f is twice continuously differentiable and $x_1^* = d^1(p, w_1, w_2) > 0$ and $x_2^* = d^2(p, w_1, w_2) > 0$ solve (ii) and that the first and second order sufficient conditions for a strict local maximum are satisfied at this point x_1^*, x_2^*. Note that the producer's supply function $y^* = s(p, w_1, w_2)$ can be determined as a function of the two input demand functions d^1 and d^2:

(iii) $s(p, w_1, w_2) \equiv f[d^1(p, w_1, w_2), d^2(p, w_1, w_2)]$.

(a) Try to determine the signs of the following derivatives: $\partial s(p, w_1, w_2)/\partial p$; $\partial d^1(p, w_1, w_2)/\partial w_1$; $\partial d^2(p, w_1, w_2)/\partial w_2$.

(b) Prove that: $\partial d^1(p, w_1, w_2)/\partial w_2 = \partial d^2(p, w_1, w_2)/\partial w_1$.

(c) Prove that: $\partial s(p, w_1, w_2)/\partial w_1 = -\partial d^1(p, w_1, w_2)/\partial p$.

Note: (b) and (c) are Hotelling symmetry conditions.
Hint: Look at the 2 first order conditions for (ii). Differentiate these 2 equations with respect to p; you will obtain a system of 2 equations involving the unknown derivatives $\partial d^1(p, w_1, w_2)/\partial p$ and $\partial d^2(p, w_1, w_2)/\partial p$. Now differentiate the 2 first order conditions with respect to w_1; you will obtain a system of 2 equations involving the derivatives $\partial d^1(p, w_1, w_2)/\partial w_1$ and $\partial d^2(p, w_1, w_2)/\partial w_1$ . . .

10. Let $F \equiv \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$ be a *symmetric* matrix that satisfies the conditions:

(i) $f_{11} < 0$;

(ii) $f_{11}f_{22} - f_{12}^2 > 0$.

Show that the following inequality holds:

(iii) $-f_{11} + 2f_{12} - f_{22} > 0$.

Hint: $-f_{11} + 2f_{12} - f_{22} = -[1, -1] \begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

11. Consider a simple two sector model for the production sector of an economy. Sector 1 (the "service" section) produces aggregate consumption C using an intermediate input M ("manufactured" goods) and inputs of labour L_1 according to the production function f:

(i) $C = f(M, L_1)$.

Sector 2 (the "manufacturing" sector produces the intermediate output M using inputs of labour L_2 according to the production function

(ii) $M = L_2$.

(Each sector can use other primary inputs such as capital, land or natural resource inputs, but since we hold these other inputs fixed in the short run, we suppress mention of them in the above notation). There is an aggregate labour constraint in the economy:

(iii) $L_1 + L_2 = \bar{L} > 0$ where $\bar{L}$ is fixed.

The manufacturer gets the revenue $p > 0$ for each unit of manufacturing output produced but the government puts a positive tax $t > 0$ on the sale of each unit of manufactures so that the service sector producer faces the price $p(1 + t)$ for each unit of M used.

The service sector producer is assumed to be a competitive profit maximizer; i.e., $M^* = M(t)$ and $L_1^* = L_1(t)$ is the solution to:

(iv) $\max_{M,L_1} \{f(M, L_1) - p(1 + t)M - wL_1\}$

where $w > 0$ is the wage rate and the price of the consumption good is 1. We assume that the following first and second order conditions for the unconstrained maximization problem (iv) are satisfied:

(v) $f_1(M^*, L_1^*) - p(1 + t) = 0;$

(vi) $f_2(M^*, L_1^*) - w = 0;$

(vii) $f_{11}^* \equiv f_{11}(M^*, L_1^*) < 0;$

(viii) $f_{22}^* \equiv f_{22}(M^*, L_1^*) < 0;$

(xi) $f_{11}^* f_{22}^* - (f_{12}^*)^2 > 0$ where $f_{12}^* \equiv f_{12}(M^*, L_1^*)$.

One more equation is required; namely we assume that the price of the manufactured good is equal to the wage rate; i.e., we have:

(x) $p = w$.

Equation (x) is consistent with profit maximizing behavior in the manufacturing sector assuming that the production function (ii) is valid.

Now substitute equations (ii), (iii) and (x) into the first order conditions (v) and (vi) and we obtain the following two equations which characterize equilibrium in this simplified economy:

(xi) $f_1(\bar{L} - L_1(t), L_1(t)] - w(t)(1 + t) = 0;$

(xii) $f_2[\bar{L} - L_1(t), L_1(t)] - w(t) = 0;$

where the 2 unknowns in (xi) and (xii) are $L_1(t)$ (employment in the service sector) and $w(t)$ (the wage rate faced by both sectors) which are regarded as functions of the manufacturer's sales tax t.

(a) Differentiate (xi) and (xii) with respect t and solve the resulting two equations for the derivatives $L_1'(t)$ and $w'(t)$.

(b) Show that $L_1'(0) > 0$. *Hint:* Use part (a) and problem 10 above.

Consumption regarded as a function of the level of sales taxation is defined as follows:

(xiii) $C(t) \equiv f[\bar{L} - L_1(t), L_1(t)]$

(c) Show that $C'(t) = -tw(t) L_1'(t)$. *Hint:* Use (xi) - (xiii).

(d) Compute $C'(0)$ and $C''(0)$. *Hint:* Use part (c).

Now we can use the derivatives in part (d) above to calculate a second order Taylor series approximation to C(t); i.e., we have

(xiv) $C(t) \approx C(0) + C'(0)t + (1/2) C''(0)t^2.$

(e) Treat (xiv) as an exact equality and show that $C(t) < C(0)$. *Hint:* Use parts (b) and (d).

Comment: This problem shows that in general, the aggregate net output of the entire production sector *falls* if transactions between sectors are taxed. There are many applications of this result. Note that (xiv) shows that the loss of output is proportional to the *square* of the tax rate, t^2.

(f) Suppose that the government now *subsidizes* the output of the manufacturing sector; i.e., t is now negative instead of being positive. Can we still conclude that $C(t) < C(0)$?

This problem shows you that you now have the mathematical tools that will enable you to construct simple models that cast some light on real life, practical economic problems.