7. Linearly Homogeneous Functions and Euler's Theorem

Let $f(x_1, \ldots, x_N) \equiv f(x)$ be a function of N variables defined over the positive orthant, $\Omega \equiv \{x: x >> 0_N\}$. Note that $x >> 0_N$ means that each component of x is positive while $x \geq 0_N$ means that each component of x is nonnegative. Finally, $x > 0_N$ means $x \geq 0_N$ but $x \neq 0_N$ (i.e., the components of x are nonnegative and at least one component is positive).

(96) **Definition:** f is (positively) linearly homogeneous iff $f(\lambda x) = \lambda f(x)$ for all $\lambda > 0$ and $x >> 0_N$.

(97) **Definition:** f is (positively) homogeneous of degree α iff $f(\lambda x) = \lambda^{\alpha} f(x)$ for all $\lambda > 0$ and $x >> 0_N$.

We often assume that production functions and utility functions are linearly homogeneous. If the producer's production function f is linearly homogeneous, then we say that the technology is subject to *constant returns to scale*; i.e., if we double all inputs, output also doubles. If the production function f is homogeneous of degree $\alpha < 1$, then we say that the technology is subject to *diminishing returns to scale* while if $\alpha > 1$, then we have *increasing returns to scale*.

Functions that are homogeneous of degree 1, 0 or -1 occur frequently in index number theory.

Recall the profit maximization problem (i) in Problem 9 above. The optimized objective function, $\pi(p, w_1, w_2)$, in that problem is called the firm's *profit function* and it turns out to be linearly homogeneous in (p, w_1, w_2).

For another example of a linearly homogeneous function, consider the problem which defines the producer's *cost function*. Let $x \geq 0_N$ be a vector of inputs, $y \geq 0$ be the output produced by the inputs x and let $y = f(x)$ be the producer's production function. Let $p >> 0_N$ be a vector of input prices that the producer faces, and define the *producer's cost function* as

(98) $$C(y, p) \equiv \min_{x \geq 0_N} \{p^T x: f(x) \geq y\}.$$

It can readily be seen, that for fixed y, C(y, p) is linearly homogeneous in the components of p; i.e., let $\lambda > 0$, $p >> 0_N$ and we have

(99) $$\begin{aligned} C(y, \lambda p) &\equiv \min_{x \geq 0_N} \{\lambda p^T x: f(x) \geq y\} \\ &\equiv \lambda \min_{x \geq 0_N} \{p^T x: f(x) \geq y\} \qquad \text{using } \lambda > 0 \\ &\equiv \lambda C(y, p). \end{aligned}$$

Now recall the definition of a linearly homogeneous function f given by (96). We have the following two very useful theorems that apply to differentiable linearly homogeneous functions.

Euler's First Theorem: If f is linearly homogeneous and once continuously differentiable, then its first order partial derivative functions, $f_i(x)$ for i = 1, 2, . . ., N, are homogeneous of degree zero and

(100) $f(x) = \sum_{i=1}^{N} x_i f_i(x) = x^T \nabla f(x)$.

Proof: Partially differentiate both sides of the equation in (96) with respect to x_i; we get for i = 1, 2, . . ., N:

(101) $f_i(\lambda x)\lambda = \lambda f_i(x)$ for all $x \gg 0_N$ and $\lambda > 0$, or

(102) $f_i(\lambda x) = f_i(x) = \lambda^0 f_i(x)$ for all $x \gg 0_N$ and $\lambda > 0$.

Using definition (97) for $\alpha = 0$, we see that equation (102) implies that f_i is homogeneous of degree 0.

To establish (100), partially differentiate both sides of the equation in (96) with respect to λ and get:

(103) $\sum_{i=1}^{N} f_i(\lambda x_1, \lambda x_2, \ldots, \lambda x_N)\partial(\lambda x_i)/\partial\lambda = f(x)$ or

$\sum_{i=1}^{N} f_i(\lambda x_1, \lambda x_2, \ldots, \lambda x_N) x_i = f(x)$.

Now set $\lambda = 1$ in (103) to obtain (100).

Q.E.D.

Euler's Second Theorem: If f is linearly homogeneous and twice continuously differentiable, then the second order partial derivatives of f satisfy the following N linear restrictions: for i = 1, . . ., N:

(104) $\sum_{j=1}^{N} f_{ij}(x)x_j = 0$ for $x \equiv (x_1, \ldots, x_N)^T \gg 0$.

The restrictions (104) can be rewritten as follows:

(105) $\nabla^2 f(x)x = 0_N$ for every $x \gg 0_N$.

Proof: For each i, partially differentiate both sides of equation (102) with respect to λ and get for i = 1, 2, . . ., N:

(106) $\sum_{j=1}^{N} f_{ij}(\lambda x_1, \ldots, \lambda x_N)\, \partial(\lambda x_j)/\partial\lambda = 0$ or

$\sum_{j=1}^{N} f_{ij}(\lambda x)x_j = 0$.

Now set □ = 1 in (106) and the resulting equations are equations (104).

Q.E.D.

Problems:

12. **[Shephard's Lemma].** Suppose that the producer's cost function $C(y, p)$ is defined by (98) above. Suppose that when $p = p^* >> 0_N$ and $y = y^* > 0$, $x^* > 0_N$ solves the cost minimization problem, so that

(i) $p^{*T}x^* = C(y^*, p^*) \equiv \min_x\{p^{*T}x: f(x) \geq y^*\}$.

(a) Suppose further that C is differentiable with respect to the input prices at (y^*, p^*). Then show that

(ii) $x^* = \nabla_p C(y^*, p^*)$.

Hint: Because x^* solves the cost minimization problem defined by $C(y^*, p^*)$ by hypothesis, then x^* must be feasible for this problem so we must have $f(x^*) \geq y^*$. Thus x^* is a feasible solution for the following cost minimization problem where the general input price vector $p >> 0_N$ has replaced the specific input price vector $p^* >> 0_N$:

(iii) $C(y^*, p) \equiv \min_x\{p^Tx: f(x) \geq y^*\}$
$\leq p^Tx^*$

where the inequality follows from the fact that x^* is a feasible (but usually not optional) solution for the cost minimization problem in (iii). Now define for each $p >> 0_N$:

(iv) $g(p) \equiv p^Tx^* - C(y^*, p)$.

Use (i) and (iii) to show that $g(p)$ is minimized (over all p such that $p >> 0_N$) at $p = p^*$. Now recall the first order necessary conditions for a minimum.

(b) Under the hypotheses of part (a), suppose $x^{**} > 0_N$ is another solution to the cost minimization problem defined in (i). Then show $x^* = x^{**}$; i.e., the solution to (i) is unique under the assumption that $C(y^*, p^*)$ is differentiable with respect to the components of p.

13. Suppose $C(y, p)$ defined by (98) is twice continuously differentiable with respect to the components of the input price vector p and let the vector $x(y, p)$ solve (98); i.e., $x(y, p) \equiv [x_1(y, p), \ldots, x_N(y, p)]^T$ is the producer's system of cost minimizing input demand functions. Define the N by N matrix of first order partial derivatives of the $x_i(y, p)$ with respect to the components of p as:

(i) $A \equiv [\partial x_i(y, p_1, \ldots, p_N)/\partial p_j] \; (\equiv \nabla_p x(y, p))$.

Show that:

(ii) $A = A^T$ and

(iii) $Ap = 0_N$.

Hint: By the previous problem, $x(y, p) \equiv \nabla_p C(y, p)$. Recall also (99) and Euler's Second Theorem.

Comment: The restrictions (ii) and (iii) above were first derived by J.R. Hicks (1939), *Value and Capital*, Appendix to Chapters II and III, part 8 and P.A. Samuelson (1947), *Foundations of Economic Analysis*, page 69. The restrictions (ii) on the input demand derivatives $\partial x_i/\partial p_j$ are known as the *Hicks-Samuelson symmetry conditions*.

So far, we have developed two methods for checking the second order conditions that arise in unconstrained optimization theory: (i) the *Lagrange-Gauss diagonalization procedure* explained in section 5 above and (iii) the *determinantal conditions method* explained in section 6 above. In the final sections of this chapter, we are going to derive a third method: the *eigenvalue method*. Before we can explain this method, we require some preliminary material on complex numbers.

8. Complex Numbers and the Fundamental Theorem of Algebra

(107) **Definition:** i is an algebraic symbol which has the property $i^2 = -1$.

Hence i can be regarded as the square root of -1; i.e., $\sqrt{-1} \equiv i$.

(108) **Definition:** A *complex number* z is a number which has the form $z = x + iy$ where x and y are ordinary real numbers. The number x is called the *real part* of z and the number y is called the *imaginary part* of z.

We can add and multiply complex numbers. To *add* two complex numbers, we merely add their real parts and imaginary parts to form the sum; i.e., if $z_1 \equiv x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

(109) $z_1 + z_2 = [x_1 + iy_1] + [x_2 + iy_2] \equiv (x_1 + x_2) + (y_1 + y_2)^i$.

To *multiply* together two complex numbers z_1 and z_2, we multiply them together using ordinary algebra, replacing i^2 by -1; i.e.,

(110) $z_1 \bullet z_2 = [x_1 + iy_1] \bullet [x_2 + iy_2]$
$\quad = x_1x_2 + iy_1x_2 + ix_1y_2 + i^2y_1y_2$

$= x_1x_2 + i^2y_1y_2 + (x_1y_2 + x_2y_1)i$
$\equiv (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i.$

Two complex numbers are *equal* iff their real parts and imaginary parts are identical; i.e., if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then $z_1 = z_2$ iff $x_1 = x_2$ and $y_1 = y_2$.

The final definition we require in this section is the definition of a complex conjugate.

(111) **Definition:** If $z = x + iy$, then the *complex conjugate* of z, denoted by $\bar{z}$, is defined as the complex number $x - iy$; i.e., $\bar{z} \equiv x - iy$.

An interesting property of a complex number and its complex conjugate is given in Problem 15 below.

Problems:

14. Let $a \equiv 3 + i$; $b \equiv 1 + 5i$ and $c \equiv 5 - 2i$. Calculate ab-c. Note that we have written $a \bullet b$ as ab.

15. Show that $z \bullet \bar{z} \geq 0$ for any complex number $z = x + iy$.

16. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers calculate $z_3 = z_1 \bullet z_2$. Show that $\bar{z}_3 = \bar{z}_1 \bullet \bar{z}_2$; i.e., the complex conjugate of a product of two complex numbers is equal to the product of the complex conjugates.

Now let f(x) be a polynomial of degree N; i.e.,

(112) $f(x) \equiv a_0 + a_1x + a_2x^2 + \ldots + a_Nx^N$ where $a_N \neq 0$,

where the fixed numbers $a_0, a_1, a_2, \ldots, a_N$ are ordinary real numbers. If we try to solve the equation $f(x) = 0$ for real roots x, then it can happen that no real roots to this polynomial equation exist; e.g., consider

(113) $1 + x^2 = 0$

so that $x^2 = -1$ and no real roots to (113) exist. However, note that if we allow solutions x to (113) to be complex numbers, then (113) has the roots $x_1 = i$ and $x_2 = -i$. In general, if we allow solutions to the equation $f(x) = 0$ (where f is defined by (112)) to be complex numbers, then there are always N roots to the equation (some of which could be repeated or multiple roots).

(114) **Fundamental Theorem of Algebra:** Every polynomial equation of the form, $a_0 + a_1x\ a_2x^2 + \ldots + a_Nx^N = 0$ (with $a_N \neq 0$) has N roots or solutions, $x_1, x_2, \ldots, x_N$, where in general, the x_i are complex numbers.

This is one of the few theorems which we will not prove in this course. For a proof, see J.V. Uspensky, *Theory of Equations*.

9. The Eigenvalues and Eigenvectors of a Symmetric Matrix

Let A be a general N by N matrix; i.e., it is not restricted to be symmetric at this point.

(115) **Definition:** λ is a *eigenvalue* of A with the corresponding eigenvector $z \equiv [z_1, z_2, \ldots, z_N]^T \neq 0_N$ iff λ and z satisfy the following equation:

(116) $Az = \lambda z; \qquad z \neq 0_N.$

Note that the eigenvector z which appears in (116) is not allowed to be a vector of zeros.

In the following theorem, we restrict A to be a symmetric matrix. In the case of a general N by N nonsymmetric A matrix, the eigenvalue λ which appears in (116) is allowed to be a complex number and the eigenvector z which appears in (116) is allowed to be a vector of complex numbers; i.e., z is allowed to have the form $z = x + iy$ where x and y are N dimensional vectors of real numbers.

(117) **Theorem:** Every N by N symmetric matrix A has N eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ where these eigenvalues are real numbers.

Proof: The equation (116) is equivalent to:

(118) $[A - \lambda I_N]z = 0_N; \qquad z \neq 0_N.$

Now *if* $[A - \lambda I_N]^{-1}$ were to exist, then we could premultiply both sides of (118) by this inverse matrix and obtain:

(119) $[A - \lambda I_N]^{-1} [A - \lambda I_N]z = [A - \lambda I_N]^{-1}\ 0_N = 0_N \qquad \text{or} \qquad z = 0_N.$

But $z = 0_N$ is not admissible as an eigenvector by definition (115). From our earlier material on determinants, we know that $[A - \lambda I_N]^{-1}$ exists iff $|A - \lambda I_N| \neq 0$. Hence, in order to hope to find a λ and $z \neq 0_N$ which satisfy (116), we *must* have:

(120) $|A - \lambda I_N| = 0.$

If N = 2, the determinantal equation (120) becomes:

(121) $0 = \left| \begin{bmatrix} a_{11}, & a_{12} \\ a_{12}, & a_{22} \end{bmatrix} - \begin{bmatrix} \lambda, & 0 \\ 0 & \lambda \end{bmatrix} \right|$

$$= \begin{vmatrix} a_{11} - \lambda, & a_{12} \\ a_{12}, & a_{22} - \lambda \end{vmatrix}$$

$$= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}^2,$$

which is a quadratic equation in λ.

In the general N by N case, if we expand out the determinantal equation (120), we obtain an equation of degree N in λ of the form $b_0 + b_1\lambda + b_2\lambda^2 + \ldots + b_N\lambda^N = 0$ and by the Fundamental Theorem of Algebra, this polynominal equation has N roots, $\lambda_1, \lambda_2, \ldots, \lambda_N$ say. Once we have found these eigenvalues λ_i, we can obtain corresponding eigenvectors $z^i \neq 0_N$ by solving

(122) $[A - \lambda_i I_N]z^i = 0_N; \qquad i = 1, 2, \ldots, N$

for a nonzero vector z^i. (We will show exactly how this can be done later).

However, both the eigenvalues λ_i and the eigenvectors z^i can have complex numbers as components in general. We now show that the eigenvalues and eigenvectors have real numbers as components when $A = A^T$.

Suppose that λ_1 is an eigenvalue of A (where $\lambda_1 = a_1 + b_1 i$ say) and $z^1 = x^1 + iy^1$ is the corresponding eigenvector. Since $z^1 \neq 0_N$, at least one component of the x^1 and y^1 vectors must be nonzero. Thus letting $\bar{z}^1 \equiv x^1 - iy^1$ be the vector of complex conjugates of the components of z^1, we have

$$\begin{aligned} z^{1T}\bar{z}^1 &= [x^{1T} + iy^{1T}][x^1 - iy1] \\ &= x^{1T}x^1 - i^2 y^{1T}y^1 - ix^{1T}y^1 + iy^{1T}x^1 \\ &= x^{1T}x^1 + y^{1T}y^1 - i[x^{1T}y^1 - y^{1T}x^1] \\ &= x^{1T}x^1 + y^{1T}y^1 \quad \text{since} \quad x^{1T}y^1 = y^{1T}x^1 \\ &= \Sigma_{i=1}^N (x_i^1)^2 + \Sigma_{i=1}^N (y_i^1)^2 \\ &> 0 \qquad (123) \end{aligned}$$

where the inequality follows since at least one of the x_i^1 or y_i^1 is not equal to zero and hence its square is positive.

By the definition of λ_1 and z^1 being an eigenvalue and eigenvector of A, we have:

(124) $Az^1 = \lambda_1 z^1$.

Since A is a real matrix, the matrix of complex conjugates of A, $\bar{A}$, is A. Now take complex conjugates on both sides of (124). Using $\bar{A} = A$ and Problem 16 above we obtain:

(125) $A\,\bar{z}^1 = \bar{\lambda}_1\,\bar{z}^1$.

Premultiply both sides of (124) by $\bar{z}^{1T}$ and we obtain the following equality:

(126) $\bar{z}^{1T}Az^1 = \lambda_1\,\bar{z}^{1T}z^1$.

Now take transposes of both sides of (126) and we obtain:

(127) $\lambda_1\,\bar{z}^{1T}\bar{z}^1 = z^{1T}A^T\bar{z}^1 = z^{1T}A\,\bar{z}^1$

where the second equality in (127) follows from the symmetry of A; i.e., $A = A^T$. Now premultiply both sides of (125) by z^{1T} and obtain:

(128) $\bar{\lambda}_1\ \bar{z}^{1T}\bar{z}^1 = z^{1T}A^T\bar{z}^1$.

Since the right hand sides of (127) and (128) are equal, so are the left hand sides so we obtain the following equality:

(129) $\lambda_1\,z^{1T}\bar{z}^1 = \bar{\lambda}_1 z^{1T}\bar{z}^1$.

Using (123), we see that $z^{1T}\bar{z}^1$ is a positive number so we can divide both sides of (129) by $z^{1T}\bar{z}^1$ to obtain:

(130) $\lambda_1 = a_1 + b_1 i = \bar{\lambda}_1 = a_1 - b_1 i$,

which in turn implies that the imaginary part of λ_1 must be zero; i.e., we find that $b_1 = 0$ and hence the eigenvalue λ_1 must be an ordinary real number.

To find a real eigenvector $z^1 = x^1 + i0_N = x^1 \neq 0_N$ that corresponds to the eigenvalue λ_1, define the N by N matrix B^1 as

(131) $B^1 \equiv A - \lambda_1 I_N$.

We know that $|B^1| = 0$ and we need to find a vector $x^1 \neq 0_N$ such that $B^1x^1 = 0_N$. Apply the Gaussian triangularization algorithm to B^1. This leads to an elementary row matrix E^1 with $|E^1| = 1$ and

(132) $E^1B^1 = U^1$

where U^1 is an upper triangular N by N matrix. Since $|B^1| = 0$, taking determinants on both sides of (132) leads to $|U^1| = 0$ and hence at least one of the N diagonal elements u_{ii}^1 of U^1 must be zero. Let $u_{i_1 i_1}^1$ be the first such zero diagonal element. We choose the components of the x^1 vector as follows: let $x_{i_1}^1$

$= 1$, let $x_j^1 = 0$ for $j > i_1$ and choose the first i_1 - 1 components of x^1 by solving the following triangular system of equations:

(133) $U^1[x_1^1, x_2^1, \ldots, x_{i_1-1}^1, 1, 0_{N-i_1}^T]^T = 0_N$.

Using the fact that the $u_{ii}^1 \neq 0$ for $i < i_1$, it can be seen that the $x_1^1, x_2^1, \ldots, x_{i_1-1}^1$ solution to (133) is unique. Hence, we have exhibited the existence of an x^1 vector such that:

(134) $U^1x^1 = 0_N$ with $x^1 \neq 0_N$.

Now premulitply both sides of (134) by $(E^1)^{-1}$ and using (132), (134) becomes

(135) $B^1x^1 = 0_N$ with $x^1 \neq 0_N$.

Obviously, the above procedure that showed that the first eigenvalue λ_1 and eigenvector x^1 for A were real can be repeated to show that all of the N eigenvalues of the symmetric matrix A are real with corresponding real eigenvectors.

Q.E.D.

Example 1: $A = [a_{11}]$; i.e., consider the case N = 1. In this case, $\lambda_1 = a_{11}$ and the eigenvector $x^1 = x_1^1$ can be any nonzero number x_1^1.

Example 2: $A = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$ i.e., A is diagonal. In this case, the determinantal equation that defines the 2 eigenvalues λ_1 and λ_2 is:

$$|A - \lambda I_2| = \begin{vmatrix} d_1 - \lambda, & 0 \\ 0, & d_2 - \lambda \end{vmatrix} = (d_1 - \lambda)(d_2 - \lambda) = 0.$$

Hence the eigenvalues of a diagonal matrix are just the diagonal elements; i.e., $\lambda_1 = d_1$ and $\lambda_2 = d_2$. Let us further suppose that the 2 diagonal elements of A are $d_1 = 1$ and $d_2 = 2$. Let us calculate the eigenvector $x^1 = (x_1^1, x_2^1)^T \neq 0_2$ that corresponds to the eigenvalue $\lambda_1 = d_1 = 1$. Define

(136) $B^1 \equiv A - \lambda_1 I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0, & 0 \\ 0, & 1 \end{bmatrix}$

In this case, B^1 is already upper triangular and the first zero diagonal element of $B^1 = U^1$ is $u_{11}^1 = 0$. In this case, we just set $x_1^1 = 1$ and $x_2^1 = 0$. It can be verified that we have $B^1x^1 = 0_2$ or $Ax^1 = \lambda_1 x^1$ with $x^1 = e^1$ and $\lambda_1 = 1$.

Now calculate the eigenvector that corresponds to the second eigenvalue of A, $\lambda_2 = d_2 = 2$. Define

$$(137) \quad B^2 \equiv A - \lambda_2 I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1, & 0 \\ 0, & 0 \end{bmatrix}$$

Also, in this case, B^2 is upper triangular, so $B^2 = U^2$ and the first zero diagonal element of U^2 is $u_{22}^2 = 0$. In this case, we set $x_2^2 = 1$ and solve

$$U^2x^2 = B^2x^2 = \begin{bmatrix} -1, & 0 \\ 0, & 0 \end{bmatrix}\begin{bmatrix} x_1^2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for $x_1^2 = 0$. Thus $x^2 = e_2$ (the second unit vector) does the job as an eigenvector for the second eigenvalue $\lambda^2 = d_2$ of a diagonal matrix.

Example 3: $A \equiv \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ The determinantal equation that defines the 2 eigenvalues of this A is

$$(138) \quad 0 = \begin{vmatrix} 1-\lambda, & 1 \\ 1, & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1$$
$$= 1 - 2\lambda + \lambda^2 - 1$$
$$= \lambda^2 - 2\lambda$$
$$= \lambda(\lambda - 2).$$

Hence the two roots of (138) are $\lambda_1 = 2$ and $\lambda_2 = 0$. To define an eigenvector x^1 for λ_1, define:

$$B^1 \equiv A - \lambda_1 I_2 = \begin{bmatrix} 1, & 1 \\ 1, & 1 \end{bmatrix} - 2\begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix} = \begin{bmatrix} -1, & 1 \\ 1, & -1 \end{bmatrix}$$

To transform B^1 into an upper triangular matrix, add the first row to the second row and we obtain U^1:

$$U^1 = \begin{bmatrix} -1, & 1 \\ 0, & 0 \end{bmatrix}$$

The first 0 diagonal element of U^1 is $u_{22}^1 = 0$. Hence set $x_2^1 = 1$ and solve

$$U^1x^1 = \begin{bmatrix} -1, & 1 \\ 0, & 0 \end{bmatrix}\begin{bmatrix} x_1^1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for $x_1^1 = 1$. Hence $x^1 = [1, 1]^T$ is an eigenvector for $\lambda_1 = 2$.