Systems of Linear Equations

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This lecture analyzes systems of linear equations. It is largely based on Chapters 6-7 of Simon and Blume.

Systems of linear appear throughout economics. There are some interesting economic models that naturally have a linear structure. Chapter 6 of Simon and Blume gives five examples: taxation and deductions, Leontief production, Markovian employment, IS-LM, and investment and arbitrage. Linear systems also arise as local approximations to nonlinear systems. Therefore, understanding linear systems is essential for understanding nonlinear systems as well.

A system of linear equations is any set of equations in which the unknown only appear linearly. An example of a linear system with 2 unknowns and 2 equations is

$$\begin{aligned} 5x_1 - 7x_2 =& 9 \\ -8x_1 + x_2 =& 0. \end{aligned}$$

In general, a linear system with m equation and n unknowns can be written

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n =& b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n =& b_2 \\ \vdots& \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n =& b_m, \end{aligned}$$

where a_{ij} and b_i are given, and x_j are unknown. This system of equations be also be written in matrix form.

$$\begin{aligned} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} =& \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \\ A\mathbf{x} =& \mathbf{b}, \end{aligned}$$

where the m by n matrix A is called the **coefficient matrix**, $\mathbf{x}$ is an $n \times 1$ vector of unknowns and $\mathbf{b}$ is an $m \times 1$ vector. We can represent a system of equations slightly more compactly by working with the **augmented coefficient matrix**,

$$\hat{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{pmatrix} = \begin{pmatrix} A\mathbf{b} \end{pmatrix}.$$

Example: Markov model of employment. Let s_t be some variable, like employment, that is randomly changing over time. We call this a random process (or stochastic process). In general, the probability of being employed at time t could depend on the entire history of $s_{t-1}, s_{t-2}, ...$. We could write it as $P(s_t|s_{t-1}, s_{t-2}, ...)$. We call s_t **Markovian** if instead the probability of being employed at time t only depends on s_{t-1}.

$$P(s_t|s_{t-1}, s_{t-2}, ...) = P(s_t|s_{t-1})$$

In this case, $\{s_t\}$ is said to be a Markov process. Given a Markov process, we are often interested in its stationary distribution. A stationary distribution is a distribution that s_t that will stay the same as time changes. If $s_t \in S$ is discrete, a stationary distribution must satisfy

$$q(s) = \sum_{s_0 \in S} P(s|s_0)q(s_0)$$

for all $s \in S$.

Markov processes appear in many areas of economics. It is usually easier to work with a Markov process than a general stochastic process. Often, we assume variables are Markovian to make a model tractable. This is really why we are about to assume employment is a Markov process. Other times, economic theory implies that some variable must follow a Markov process. This sometimes happens with asset prices. Markov processes are also very important for Bayesian estimation.

If employment follows a Markov process, then its evolution over time is completely described by four probabilities: the probability that someone who is employed today is employed tomorrow, the probability that someone who is employed today is unemployed tomorrow, the probability that someone who is unemployed today is employed tomorrow, and the probability that someone who is unemployed today is also unemployed tomorrow. We will denote these four probabilities by p_{ee}, p_{ue}, p_{eu}, and p_{uu}. Given these for probabilities, we might be interested in the equilibrium employment rate, i.e. the employment rate in the stationary distribution of the process. Let π_e and π_u be stationary employment and unemployment rates. They must satisfy

$$\begin{aligned}\pi_e =& p_{ee}\pi_e + p_{eu}\pi_u \\ \pi_u =& p_{ue}\pi_e + p_{uu}\pi_e \\ 1 =& \pi_e + \pi_u,\end{aligned}$$

or, equivalently, in the general form written above

$$\begin{aligned}(p_{ee} - 1)\pi_e + p_{eu}\pi_u =& 0 \\ p_{ue}\pi_e + (p_{uu} - 1)\pi_e =& 0 \\ \pi_e + \pi_u =& 1.\end{aligned}$$

The augmented matrix for this system is

$$\hat{A} = \begin{pmatrix} p_{ee} - 1 & p_{eu} & 0 \\ p_{ue} & p_{uu} - 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Three questions to ask about such a system of equations are:

(1) Does any solution exist?
(2) How many solutions exist?
(3) How can a solution be computed?

We will begin by examining the first question. Then, we will see that the answers to the first two questions depend on the coefficients of the system of equations.

1. Solving systems of equations

You likely already have experience solving small systems of equations. The two basic techniques are

(1) substitution: solve for one variable in terms of the others and substitute
(2) elimination: add a multiple of one equation to another to eliminate one variable.

One way of viewing elimination is that transforms one system of equations to another that is easier to solve, while ensuring the solution remains the same. Three basic **equation operations** that we can perform while preserving the solution of the system are:

(1) Multiply an equation by a non-zero constant,
(2) Add a multiple of one equation to another, and
(3) Interchange two equations.

We could also perform these operations on the augmented coefficient matrix. We then call them **row operations** instead. Given a reasonably small system of equations, you might be able to solve the system without thinking too carefully about the steps involved. However, if we want to solve large systems of equations (or write a computer programmer to solve large systems of equations), we will need to think carefully about the steps involved.

1.1. Gaussian elimination.

Gaussian elimination is the process of using these operations to transform the augmented matrix of a system of equation into row echelon form. A matrix is in **row echelon form** if each row begins with more zeros than the row above it or the row is all zeros. The first non-zero entry in each row of a matrix in row echelon form is called a **pivot**. **Gaussian elimination** can be performed as follows:

(GE1) Identify the first column to contain any non-zero elements, call this column c^*.
(GE2) Interchange rows so that a nonzero entry appears at the top of column c^*.
(GE3) Add a multiple of the first row to each of the rows below so that the entries in column c^* below the first row are zero.
(GE4) Repeat GE1-GE2 on the submatrix consisting of the lower right part of the original matrix below the first row and to the right of column c^*. Stop if this submatrix has no columns or has no rows.

Example 1.1 (Gaussian elimination)**.** Consider the system

$$\begin{array}{rcrcrcrcr} & & 3x_2 & + & 2x_3 & - & 4x_4 & = & 4 \\ 6x_1 & - & x_2 & & & + & x_4 & = & -2 \\ x_1 & + & x_2 & + & x_3 & & & = & 1 \\ & & & & 4x_3 & - & x_4 & = & 3 \end{array}$$

The augmented matrix for this system is

$$\hat{A} = \begin{pmatrix} 0 & 3 & 2 & -4 & 4 \\ 6 & -1 & 0 & 1 & -2 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 4 & -1 & 3 \end{pmatrix}$$

Following the steps above:

GE1 $c^* = 1$

GE2 Swap 2nd and 1st row.

$$\begin{pmatrix} 6 & -1 & 0 & 1 & -2 \\ 0 & 3 & 2 & -4 & 4 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 4 & -1 & 3 \end{pmatrix}$$

GE3 Add $-1/6(row\ 1)$ to row 3.

$$\begin{pmatrix} 6 & -1 & 0 & 1 & -2 \\ 0 & 3 & 2 & -4 & 4 \\ 0 & 7/6 & 1 & -1/6 & 4/3 \\ 0 & 0 & 4 & -1 & 3 \end{pmatrix}$$

GE1 Now ignoring first row and column, $c^* = 2$.

GE2 Leave row 2 where it is.

GE3 Add $-7/18$ row 2 to row 3.

$$\begin{pmatrix} 6 & -1 & 0 & 1 & -2 \\ 0 & 3 & 2 & -4 & 4 \\ 0 & 0 & 2/9 & 25/18 & -4/9 \\ 0 & 0 & 4 & -1 & 3 \end{pmatrix}$$

GE1 Now ignoring first to columns and rows, $c^* = 3$.

GE2 Leave row 3.

GE3 Add -18 row 3 to row 4.

$$\begin{pmatrix} 6 & -1 & 0 & 1 & -2 \\ 0 & 3 & 2 & -4 & 4 \\ 0 & 0 & 2/9 & 14/9 & -4/9 \\ 0 & 0 & 0 & -26 & 11 \end{pmatrix}$$

Given the above row echelon form, it is relatively easy to solve the system of equations. From the last row, we know $x_4 = -11/26$. Substituting into the second last row, we get

$x_3 = 9/2(-4/9 + 14/9 \times 11/26)$. Back substituting back to the first row gives a complete solution.

It is always possible to transform a matrix into row echelon form. Moreover, we can prove that the above procedure always work.

Theorem 1.1 (Existence of row echelon form)**.** *Any matrix can be put into row echelon form using Gaussian elimination.*

Proof. Let m be the number of rows of a matrix and n be the number of columns. We will prove the theorem by induction on m and n.

Any 1 by n matrix is already in row echelon form.

Also, given any m by 1 matrix, if it is all zeros, it is already in row-echelon form. Otherwise, it contains a nonzero entry. We can move this nonzero entry to the first row. Then we can add a multiple of the first entry to all other entries to make all entries after the first into zeros. Thus, any m by 1 matrix can be put into row echelon form.

For example, consider

$$
\begin{aligned}
A &= \begin{pmatrix} 0 \\ 2 \\ -7 \\ 9 \end{pmatrix} \\
&\quad \text{swap row 1 and 2} \\
&\simeq \begin{pmatrix} 2 \\ 0 \\ -7 \\ 9 \end{pmatrix} \\
&\quad \text{add row 1 times 7/2 to row 3} \\
&\simeq \begin{pmatrix} 2 \\ 0 \\ 0 \\ 9 \end{pmatrix} \\
&\quad \text{add row 1 times -9/2 to row 3} \\
&\simeq \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&\quad \text{It's now in row echelon form}
\end{aligned}
$$

Now suppose we can put matrix with less than m rows or less than n columns into row echelon form. Given an m by n matrix either the first column contains a nonzero entry or it does not. If the first column is all zeros, then we may ignore it, and we just have to transform the remainging m by $n-1$ matrix into row echelon form, which can be done due to our inductive assumption.

If the first column is not all zeros, then we can follow steps (GE1-GE3) to make the first element of the matrix nonzero and all other entries in the first column zero. We can then just work on transforming the remaining $m-1$ by n matrix after the first row into row echelon form, which again is possible by our inductive assumption. □

Note that the row echelon form of a matrix is not unique because, for example, we could multiply any row by a constant and the matrix would still be in row echelon form.

1.2. **Gauss-Jordan elimination.** A matrix is in **reduced row echelon form** if it is in row echelon form with each pivot equal to one and each column that contains a pivot has no other non-zero entries. For example,

$$\begin{pmatrix} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{pmatrix}$$

is in reduced row echelon form. The solution to a system of linear equations is given immediately by its augmented matrix in reduced row echelon form. In the previous example, the solution is $x_1 = b_1$, $x_2 = b_2$, and $x_3 = b_3$. **Gauss-Jordan elimination** transforms a matrix into reduced row echelon form and can be performed as follows:

(GJ1) Put the matrix into row echelon form by performing Gaussian elimination
(GJ2) Divide the bottom row by its pivot.
(GJ3) Add a multiple of the bottom row to each row above it such that the column above the bottom row's pivot is made equal to all zeros.
(GJ4) Repeat GJ2 and GJ3 with the next row up.

Example 1.2 (Gauss-Jordan elimination)**.** Suppose we have performed Gaussian elimination to get

$$\begin{pmatrix} 2 & 3 & 0 & -1 \\ 0 & -1 & -2 & 3 \\ 0 & 0 & 7 & 14 \end{pmatrix}$$

Following the steps above, we get

$$\begin{pmatrix} 2 & 3 & 0 & -1 \\ 0 & -1 & -2 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix} (\text{GJ2})$$

$$\begin{pmatrix} 2 & 3 & 0 & -1 \\ 0 & -1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{pmatrix} (\text{GJ3})$$

$$\begin{pmatrix} 2 & 3 & 0 & -1 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 2 \end{pmatrix} (\text{GJ2})$$

$$\begin{pmatrix} 2 & 0 & 0 & 20 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 2 \end{pmatrix} (\text{GJ3})$$

$$\begin{pmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 2 \end{pmatrix} (\text{GJ2})$$

As with row echelon form and Gaussian elimination, we can prove that a reduced row echelon form always exists and Gauss-Jordan elimination can produce it.

Theorem 1.2 (Existence of reduced row echelon form)**.** *Any matrix can be put into reduced row echelon form using Gauss-Jordan elimination.*

Proof. Let A by an m by n matrix. By Theorem 1.1, A can be transformed into row echelon form. Steps GJ2-GJ4 will transform A into reduced row echelon form. □

2. Existence of solutions

We now have a method for solving systems of equations. Will this method always work? The answer is no. It is easy to write down systems of equations that cannot be solved. For example, $\begin{matrix} x & = & 2 \\ -x & = & 3 \end{matrix}$, has no solutions. However, it is not always so obvious when a system of equations has no solutions.

Example 2.1. Consider the system:

$$\begin{aligned} x + 2y - z =& 2 \\ 4y + z =& 5 \\ -2x - 4y + 2z =& 1. \end{aligned}$$

Let's transform this system into row echelon form. Let $\hat{A} \simeq \hat{B}$ mean that the systems of equations represent by $\hat{A}$ and $\hat{B}$ have the same solution.

$$\begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & 4 & 1 & 5 \\ -2 & -4 & 2 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & 4 & 1 & 5 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

The third equation in the transformed system is

$$0x + 0y + 0z = 3,$$

which has no solution. Then, the entire transfrom system must have no solution. By construction, the transformed system has the same solutions as the original system, so the original system of equations must also have no solution.

In the preceding example, we saw a system of equations with no solution because its row echelon form had a row with all zeros except for the final column. This observation applies more generally.

Definition 2.1. The **rank** of a matrix is the number of nonzero rows in its row echelon form.

This definition is slighty problematic because, as stated earlier, the row echelon form of a matrix is not unique. To show that rank is well defined, we should prove that any row echelon form of a matrix has the same number of nonzero rows. We will prove that rank is a well-defined a little later in the course.

Lemma 2.1. *The rank of a matrix A is always less than or equal to the number of columns of A and less than or equal to the number of rows of A.*

Proof. The first claim follows form the definition of row echelon form. If each row of the row echelon form of A must start with more zeros than the preceding row, then there can be at most as many nonzero rows as there are columns. A also cannot have more nonzero rows than total rows, so the second claim is trivial. $\square$

Lemma 2.2. *Let A be a coefficient matrix and $\hat{A}$ be an augmented coefficient matrix. Then* $\operatorname{rank} A \leq \operatorname{rank} \hat{A}$.

Proof. Let

$$\hat{A}' = \begin{pmatrix} a'_{11} & \cdots & a'_{1n} & b'_1 \\ \vdots & & & \vdots \\ a'_{m1} & \cdots & a'_{mn} & b'_m \end{pmatrix}$$

be a row echelon form of $\hat{A}$. Then the first n columns of $\hat{A}'$ is a row echelon form for A. Finally, the number of nonzero rows of $\hat{A}'$ must be greater than or equal to the number of nonzero rows of its submatrix,

$$\begin{pmatrix} a'_{11} & \cdots & a'_{1n} \\ \vdots & & \vdots \\ a'_{m1} & \cdots & a'_{mn} \end{pmatrix}.$$

$\square$

Theorem 2.1 (Existence of solutions)**.** *A system of linear equations with coefficient matrix A and augmented coefficient matrix $\hat{A}$ has a solution (perhaps more than one) if and only if* $\operatorname{rank} A = \operatorname{rank} \hat{A}$.

Proof. We'll first prove the "only if" part of this theorem. Suppose rank$A \neq$ rank$\hat{A}$. We want to show that then there are no solutions to the associated system of equations. From 2.2, we know that rank$A <$ rank$\hat{A}$. There is a zero row in the row echelon form of A and a corresponding nonzero row in the row echelon form of $\hat{A}$. The equation associated with this row is of the form

$$0x_1 + ... + 0x_n = b'_m$$

for some $b'_m \neq 0$. As in the example, this equation has no solution, so the system has no solution.

Now we will prove the "if" part of the theorem. Suppose rank$A =$ rank$\hat{A}$. Let $\hat{A}'$ be a row echelon form of $\hat{A}$ and $A' =$ the first n columns of $\hat{A}'$ be the associated row echelon form of A. We can prove the existence of solutions by induction on the number of rows. If A' is 1 by n with a nonzero entry, say a_{1j}, we can produce a solution by choosing any values for $\{x_1, ..., x_n\} \setminus \{x_j\}$ and set

$$x_j = \frac{1}{a_{1j_1}} \left(b_1 - \sum_{k \neq j} a_{1k} x_k \right).$$

Now let A be m by n and suppose we have proven the claim for all $(m-1)$ by n matrices. If the mth row of A' has all zeros, then we may ignore it and only work with the first $m-1$ rows. If the mth row of A' has a nonzero entry, let a'_{mj} be the first nonzero entry in the mth row. Choose any values for $x_{j+1}, ..., x_n$ and set

$$x_j = \frac{1}{a'_{mj_1}} \left(b'_m - \sum_{k=j+1}^{n} a'_{mk} x_k \right).$$

Substitute these values of $x_j, ..., x_n$ into the $m-1$ rows above to produce the system

$$\hat{B} = \begin{pmatrix} a'_{1,1} & \cdots & a'_{1,j-1} & b'_1 - \sum_{k=j}^{n} a'_{1,k} x_k \\ 0 & \ddots & & \vdots \\ & \cdots & a'_{m-1,j-1} & b'_{m-1} - \sum_{k=j}^{n} a'_{m-1,k} x_k \end{pmatrix}.$$

We must have one of the elements of the $m-1$ row non zero because otherwise $\hat{A}'$ would not have been in row echelon form. Similarly, all of $\hat{B}$ must be row echelon form because $\hat{A}$ was in row echelon form. Thus, rank$\hat{B} = m-1$ and the coefficient matrix associated with $\hat{B}$ also has rank $m-1$. By the inductive assumption, there exists $x_1, ..., x_{j-1}$ that solve the system represented by $\hat{B}$. This solution combined with the values for $x_j, ..., x_n$ described above is a solution to the entire original system. $\square$

We now have a nice condition for when there exists at least one solution to a system of linear equations. If there is a solution can be more than one? From looking at simple systems like $x + y = 0$, the answer is clearly yes. If you look carefully at the proof of Theorem 2.1 you might be able to see that multiple solutions will exist whenever at least one solution exists and the row echelon form of the coefficient matrix has a row with more than one more zero than the row preceding it. That is, there will be multiple solutions whenever the rank of the augmented coefficient matrix is equal to rank of the coefficient matrix, and rank is less than the number of variables in the system (which is equal to the number of columns). We will state these ideas formally and prove them below. Before that, let's look at another example.

Example 2.2. Consider the system:

$$\begin{aligned} 4y + z =& 5 \\ x + 2y - z =& 2 \\ -8y - 2z =& -10. \end{aligned}$$

Let's transform this system into row echelon form. Let $\hat{A} \simeq \hat{B}$ mean that the systems of equations represent by $\hat{A}$ and $\hat{B}$ have the same solution.

$$\begin{aligned} \begin{pmatrix} 0 & 4 & 1 & 5 \\ 1 & 2 & -1 & 2 \\ 0 & -8 & 2 & -10 \end{pmatrix} \simeq& \begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & 4 & 1 & 5 \\ 0 & -8 & 2 & -10 \end{pmatrix} \\ \simeq& \begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & 4 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

For any value of $z \in \mathbb{R}$, $y = \frac{1}{4}(5 - z)$ and $x = 2 - \frac{1}{2}(5 - z) + z = -\frac{1+z}{2}$ is a solution to the system.

When solving a system of equations, we call variables whose value is indeterminate **free** variables. We call variables whose value is either completely determined or determined by the value of the free variables **basic**. In the above examples, z is a free variable and x and y are basic. Note that free and basic are just names that are sometimes useful, but they are not concrete definitions. In the above example, we could have just as easily described the set of solutions by saying: $x \in \mathbb{R}$, $z = 2x - 1$, and $y = \frac{3-x}{2}$.

In the above example we saw a system of equations with infinitely many solutions. It turns out that whenever there is more than one solution, there must be infinitely many.

Lemma 2.3. [1] *Suppose $\mathbf{x}_1$ and $\mathbf{x}_2$ are two distinct solutions to the system of equations $A\mathbf{x} = \mathbf{b}$. Then the system of equations has (uncountably) infinitely many solutions.*

Proof. Let $w \in \mathbb{R}$. Consider $\mathbf{x}(w) = w\mathbf{x}_1 + (1 - w)\mathbf{x}_2$. Since $\mathbf{x}_1 \neq \mathbf{x}_2$, for each w, $\mathbf{x}(w)$ is unique. Also,

$$\begin{aligned} A\mathbf{x}(w) =& A\left(\mathbf{x}_1 w + \mathbf{x}_2(1 - w)\right) \\ =& A\mathbf{x}_1 w + A\mathbf{x}_2(1 - w) \\ =& bw + b(1 - w) \\ =& b, \end{aligned}$$

so $\mathbf{x}(w)$ is another solution. □

[1] We state and prove this lemma using matrix notation. We will study matrix algebra in greater detail in the next lecture. If you are uncomfortable with matrix notation and operations here, it may help to restate and prove the theorem without using matrices.

Another interesting observation about the previous example is that if were to just change b_3 from -10 to any other number, it would have no solutions instead of a unique solution. In particular, the third row of the row echelon form would be $\begin{pmatrix} 0 & 0 & 0 & b_3+10 \end{pmatrix}$. It would be good to know when the existence of solutions depends on the choice of b and when it does not.

Theorem 2.2 (Solution existence)**.** *A system of linear equations with coefficient matrix A will have a solution for any choice of $b_1, ..., b_m$ if and only if* $\operatorname{rank} A$ *is equal to the number of rows of A.*

Proof. If rank of A is equal to the number of rows of A, then the row echelon form of A has no all zero rows. For any choice of b, the augmented matrix, $\hat{A}$, must also have no all zero rows. Hence, $\operatorname{rank} A = \operatorname{rank} \hat{A}$ and by theorem 2.1, at least one solution exists.

If $\operatorname{rank} A$ is less than the number of rows of A, then the last row of the row echelon form of A, A' has all zeros. If we augment produce an augmented matrix in row echelon form $\hat{A}'$ with $b_m \neq 0$, then the last equation has no solutions. We can then reverse the steps of Gaussian elimination used to produce A' to arrive at an augmented matrix $\hat{A}$ corresponding to a system with coefficient matrix A that has no solution. $\square$

The following corollary is an immediate consequence of the second part of the above proof.

Corollary 2.1. *For any system of equations with more equations than variables, there exists a choice of* $\boldsymbol{b}$ *such that no solutions exist.*

We call a system of equations with more equations than variables **overdetermined**. A system with more variables than equations is called **underdetermined**.

3. Uniqueness of solutions

Theorem 3.1 (Solution uniqueness)**.** *Any system of equations with coefficient matrix A has at most one solution for any $b_1, ..., b_m$ if and only if* $\operatorname{rank} A$ *equals the number of columns of A.*

Proof. Suppose $\operatorname{rank} A$ is equal to the number of columns of A. Then the row echelon form of A must be of the form

$$\begin{pmatrix} a'_{11} & a'_{12} & ... & a'_{1n} \\ 0 & a'_{22} & ... & a'_{2n} \\ 0 & 0 & \ddots & \\ 0 & \cdots & 0 & a'_{kn} \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

The only possible solution is $x_n = b_k / a'_{kn}$, $x_{n-1} = \frac{b_{k-1} - a'_{k-1,n} x_n}{a'_{k-1,n-1}}$. Thus, if any solution exists, it is unique.

Conversely, suppose $\operatorname{rank} A$ is less than the number of columns of A. We can prove that A can have multiple solutions by performing induction on m. If A is 1 by n and has at least one non-zero entry, then the system will have infinite solutions for any b_1. On the other hand, if A is all zeros, then the system has infinite solutions for $b_1 = 0$.[2]

Now, suppose we have shown that any $m-1$ by n matrix with rank less than the number of columns can have multiple solutions. If A is m by n with rank less than n, consider three cases: (i) the last row of the row echelon form of A is identically zero, (ii) the last row of the row echelon from of A has one non-zero entry and (iii) the last row of the row echelon form of A has multiple non-zero entries. Every A will fit into one of these three cases.

In case (iii), we can produce a solution for any value of x_n, so multiple solutions exist.

In case (i), we can delete the last row of the row echelon form of A without changing its rank. Let A_1 be the row echelon form of A with the last row deleted. When $b_m = 0$, any solution to A_1 is also a solution to A. Furthermore, the rank of A_1 must equal the rank of A, and the number of columns of A_1 equals the number of columns of A. Hence, A_1 is $m-1$ by n with rank less than the number of columns, so by induction, multiple solutions will exist.

In case (ii), let A_1 be the row echelon form of A with last row and last column deleted. Finally, $\text{rank}A_1 = \text{rank}A - 1$ and A_1 has one column less than A, so A_1 is $m-1$ by $n-1$ and by induction multiple solutions exist. Let $b_1, ..., b_{m-1}$ combined with A_1 produces multiple solutions. Then any solution to the system A_1 along with $x_n = b_m/\tilde{a}_{mn}$, where $\tilde{a}_{mn}$ is the last entry in the last row of the row echelon form of A, will be a solution to the system with coefficients A and right hand side $b_1 + \tilde{a}_{1n} b_m/\tilde{a}_{mn}, \cdots, b_{m-1} + \tilde{a}_{m-1,n} b_m/\tilde{a}_{mn}, b_m$. Thus, multiple solutions to A also exist. □

The following corollary is a nice consequence of the above proof.

Corollary 3.1. *If* rank$A$ *is less than the number of columns of A then either no solutions exists or multiple solutions exists.*

Proof. As noted in the footnote in the previous proof, the first step showed that any 1 by n matrix with rank less than n has either no solutions or multiple solutions. The same inductive argument as in the previous proof then shows that whenever any solution exists, there must be multiple solutions. □

We call a coefficient matrix A **nonsingular** if for any $b_1, ..., b_m$ the system of equations has exactly one solution. Combining the last two theorems, (2.2 and 3.1) we get the following corollary.

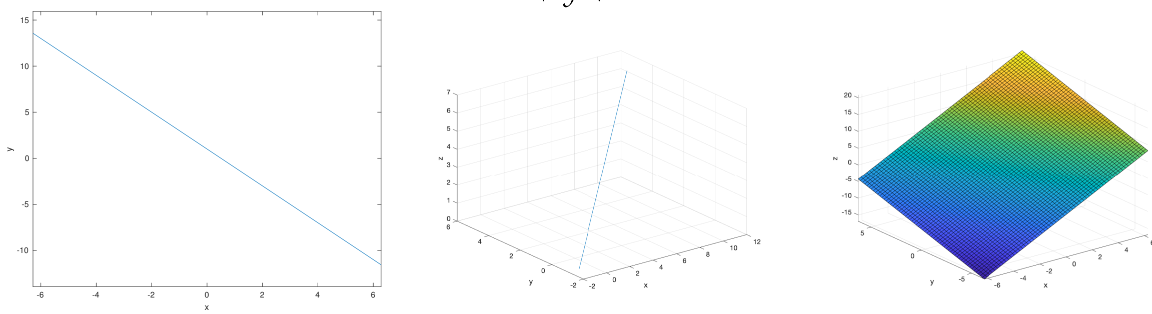
Corollary 3.2. *A is nonsingular if and only if A has an equal number of columns and rows (A is square) and has rank equal to its number of columns (or rows).*

We now know conditions under which a system has a solution and when the solution is unique. To review, we know from theorem 2.1 that a particular system has a solution if and only if $\text{rank}A = \text{rank}\hat{A}$. Additionally, if $\text{rank}A$ is equal to its number of columns,

[2]The only other possible case is that A is all zeros and $b_1 \neq 0$, in that case A has no solutions. Thus, if A has any solution, then A has multiple solutions. This observation is not needed for this proof, but will be used to prove corollary 3.1.

FIGURE 1. Solution sets

$-2x + y = 1$ $\qquad$ $\begin{aligned} x - y - z &= 0 \\ -2x + y + 3z &= 1 \end{aligned}$ $\qquad$ $-2x - y + z = 2$



then the solution is unique (theorem 3.1). On the other hand if rank$A$ is less than the number of columns, then there are infinite solutions (corollary 3.1 and lemma 2.3).

4. Set of solutions

A final issue to investigate is: if there are infinite solution, how can we describe the set of solutions. Figure 1 plots the solution sets to three systems of equations with multiple solutions. Generalizing the these three examples, we can guess that:

- The set of solutions to an equation with two variables,
$$ax + by = c,$$
is a line in $\mathbb{R}$.
- The set of solutions to two equation with three variables,
$$\begin{aligned} a_1 x + b_1 y + c_1 z =& d_1 \\ a_2 x + b_2 y + c_2 z =& d_2, \end{aligned}$$
is also a line in $\mathbb{R}^3$.
- The set of solutions to a single equation with three variables,
$$a_1 x + b_1 y + c_1 z = d_1$$
is a plane in $\mathbb{R}^2$.

We will prove that these three guesses are true next week. In fact, we will prove that when generalized to systems with more variables, these guesses remain true. In this lecture we will just state the general result. To do so, we need some more definitions. We will use the definition much more when we talk about Euclidean spaces in a week or so.

Definition 4.1. The set $S \subseteq \mathbb{R}^n$ is called a **linear subspace** if it is closed under (i) scalar multiplication and (ii) addition in other words, if

(i) for every $(x_1, ..., x_n) \in S$ and $a \in \mathbb{R}$, we have $(ax_1, ax_2, ..., ax_n) \in S$, and
(ii) for every $(x_1, ..., x_n) \in S$ and $(y_1, ..., y_n) \in S$, we have $(x_1 + y_1, ..., x_n + y_n) \in S$

An implication of (i) is that $0 \in$ any linear subspace (take $a = 0$). Thus, linear subspaces are lines, planes, and hyperplanes passing through the origin.

Definition 4.2. A set of vectors in $\mathbb{R}^n$, $\{\mathbf{x}_j = (x_1^j, ..., x_n^j)\}_{j=1}^J$, is **linearly independent** if the only solution to

$$\sum_{j=1}^{J} c_j \mathbf{x}_j = 0$$

is $c_1 = c_2 = ... = c_J = 0$.

Observe that two points on a line passing through 0 are not linearly independent. Three points on a plane that contains $(0, 0)$ are also not linearly independent. This suggests the following definition.

Definition 4.3. The dimension of a linear subspace $S \subseteq \mathbb{R}^n$ is the cardinality of the largest set of linearly independent elements in S.

With this defition, a line has dimension one, a plane has dimension two, and $\mathbb{R}^n$ has dimension n. We can now state a result that summarizes everything we know about the solutions to linear systems.

Theorem 4.1 (Rouché-Capelli)**.** *A system of linear equations with n variables has a solution if and only if the rank of its coefficient matrix, A, is equal to the rank of its augmented matrix, $\hat{A}$. If a solution exists and* rank$A$ *is equal to its number of columns, the solution is unique. If a solution exists and* rank$A$ *is less than its number of columns, there are infinite solutions. In this case the set of solutions is of the form*[3]

$$\{s + x^* \in \mathbb{R}^n : s \in S \text{ and } Ax^* = b\}$$

where S is the linear subspace of dimension $n -$ rank$A$ defined by $S = \{s \in \mathbb{R}^n : As = 0\}$ and x^* is any single solution to $Ax = b$.

The first part of this theorem is just a restatement of things we have already proven. We will prove the last claim, that the set of solutions forms a linear subspace of dimension $n -$ rank$A$, next week.

Example: Markov model of employment (continued). We can now answer the three questions posed at the beginning of this lecture.

[3]A set of this form is called an affine subspace. It is a linear subspace that has been shifted so that it no longer necessarily contains the origin.

(1) Does any solution exist?
A solution exists if $\operatorname{rank} A = \operatorname{rank} \hat{A}$. For this example,

$$A = \begin{pmatrix} p_{ee}-1 & p_{eu} \\ p_{ue} & p_{uu}-1 \\ 1 & 1 \end{pmatrix}.$$

$$\hat{A} = \begin{pmatrix} p_{ee}-1 & p_{eu} & 0 \\ p_{ue} & p_{uu}-1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

If we perform Gaussian elimination on A, we get[4]

$$\begin{pmatrix} p_{ee}-1 & p_{eu} \\ p_{ue} & p_{uu}-1 \\ 1 & 1 \end{pmatrix} \simeq \begin{pmatrix} p_{ee}-1 & p_{eu} \\ 0 & \frac{p_{ee}-1-p_{eu}}{p_{ee}-1} \\ 0 & \frac{(p_{ee}-1)(p_{uu}-1)-p_{eu}p_{ue}}{p_{ee}-1} \end{pmatrix}$$
$$\simeq \begin{pmatrix} p_{ee}-1 & p_{eu} \\ 0 & \frac{p_{ee}-1-p_{eu}}{p_{ee}-1} \\ 0 & 0 \end{pmatrix}.$$

For $\hat{A}$, we get

$$\begin{pmatrix} p_{ee}-1 & p_{eu} & 0 \\ 1 & 1 & 1 \\ p_{ue} & p_{uu}-1 & 0 \end{pmatrix} \simeq \begin{pmatrix} p_{ee}-1 & p_{eu} & 0 \\ 0 & \frac{p_{ee}-1-p_{eu}}{p_{ee}-1} & 1 \\ 0 & \frac{(p_{ee}-1)(p_{uu}-1)-p_{eu}p_{ue}}{p_{ee}-1} & 0 \end{pmatrix}$$
$$\simeq \begin{pmatrix} p_{ee}-1 & p_{eu} & 0 \\ 0 & \frac{p_{ee}-1-p_{eu}}{p_{ee}-1} & 1 \\ 0 & 0 & -\frac{(p_{ee}-1)(p_{uu}-1)-p_{eu}p_{ue}}{p_{ee}-1}\frac{p_{ee}-1}{p_{ee}-1-p_{eu}}. \end{pmatrix}$$

and we see that the rank $\hat{A}$ will be greater than the rank A of unless

$$-\frac{(p_{ee}-1)(p_{uu}-1)-p_{eu}p_{ue}}{p_{ee}-1}\frac{p_{ee}-1}{p_{ee}-1-p_{eu}} = 0.$$

Fortunately, if these are valid probabilities, then $p_{ee}+p_{ue}=1$ and $p_{uu}+p_{eu}=1$, so

$$\begin{aligned}(p_{ee}-1)(p_{uu}-1)-p_{eu}p_{ue} =& (-p_{ue})(-p_{eu})-p_{eu}p_{ue} \\ =& 0.\end{aligned}$$

Thus, the system does have a solution.

(2) How many solutions exist?
From the above, we see that $\operatorname{rank} A = 2$ provided that $-p_{ue}-p_{eu} \neq 0$ and $p_{ee}-1 \neq 0$. Therefore, there is a unique solution.

[4]You should check that the steps being performed do not involve division by zero.